Stochastic Modeling for Two-Stage and Multivariate Degradation Processes

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Outline

1 Introduction

2 Two-phase degradation model

3 Multivariate degradation model

4 Conclusion

Failure data

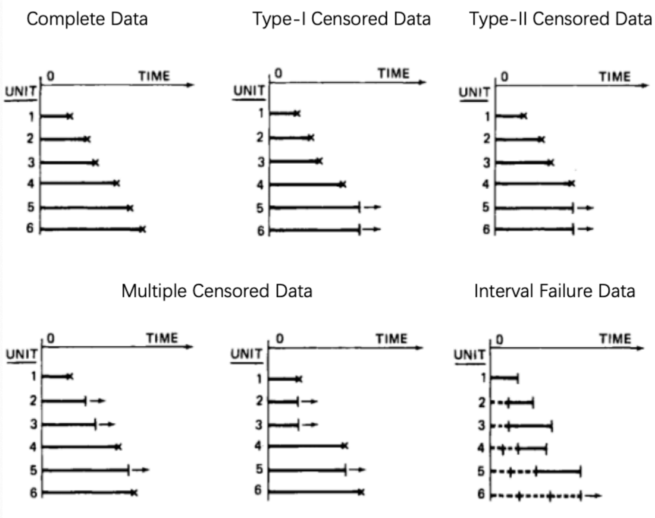


Figure 1: Types of failure data.

Accelerated life testing

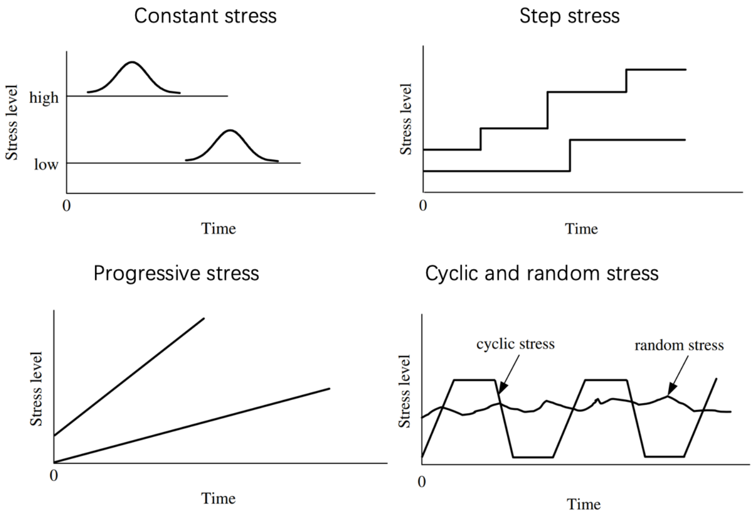
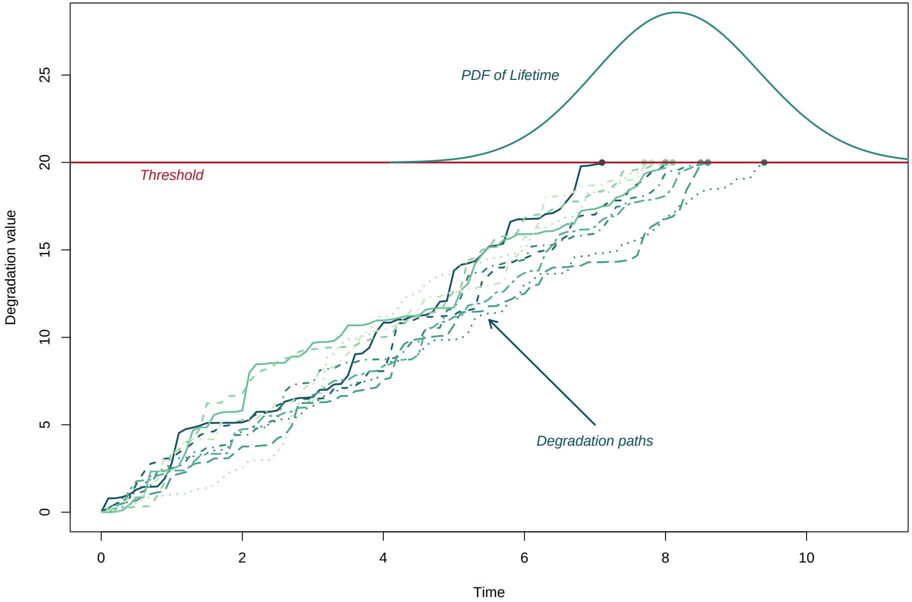
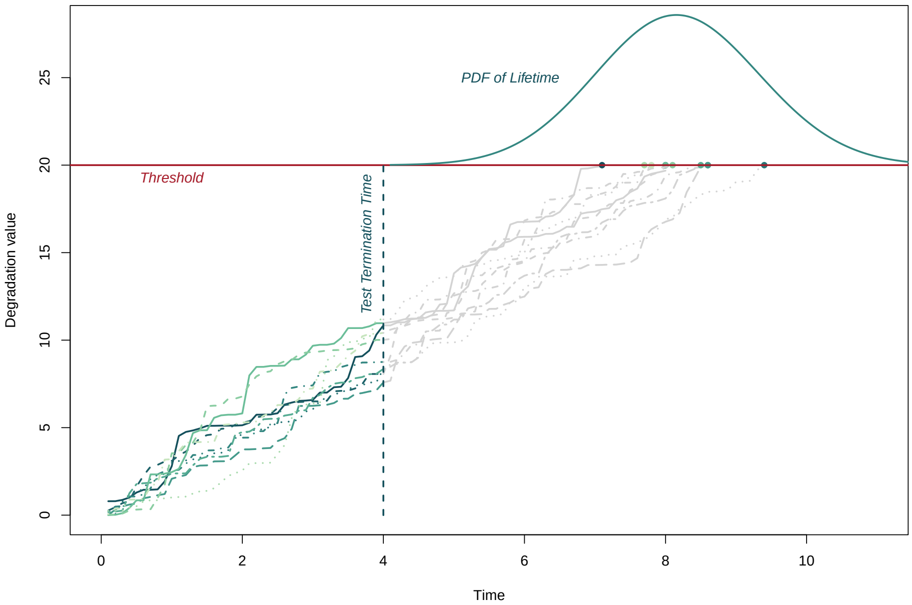


Figure 2: Types of accelerated life testing.

Degradation data

- In modern society, many products are designed with high reliability, and failure data are hard to collected for these products, even using accelerated life testing.
- Degradation data provide a useful resource for obtaining reliability information for highly reliable products.
  - Loss of light output from an LED array
  - Power output decrease of photovoltaic arrays
  - Corrosion in a pipeline
  - Vibration from a worn bearing in a wind turbine
  - Loss of gloss and colour of an automobile finish
  - ...


PDF of Lifetime
Threshold
Degradation paths
Degradation value
Time
0
5
10
15
20
25
0
2
4
6
8
10


PDF of Lifetime
Threshold
Test Termination Time
Degradation value
Time
0
5
10
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25
0
2
4
6
8
10

- Let $Y(t)$ be the degradation process of the performance characteristic (PC), and ω be the failure threshold level.
- Define that the lifetime of product $T = \inf\{t : Y(t) \geq \omega\}$.

Degradation models

- General degradation path models

$$Y(t) = D(t|\boldsymbol{\beta}, \boldsymbol{b}) + \epsilon.$$

- Stochastic degradation models, i.e., Wiener process (Liao and Tseng, 2006), gamma process (Park and Padgett, 2005), inverse Gaussian process (Wang and Xu, 2010), exponential dispersion process (Zhou and Xu, 2019), variance gamma, Ornstein–Uhlenbeck, etc.
- Two review papers: Ye and Xie (2015), Zhang et al. (2018).

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Motivated example: OLED degradation data

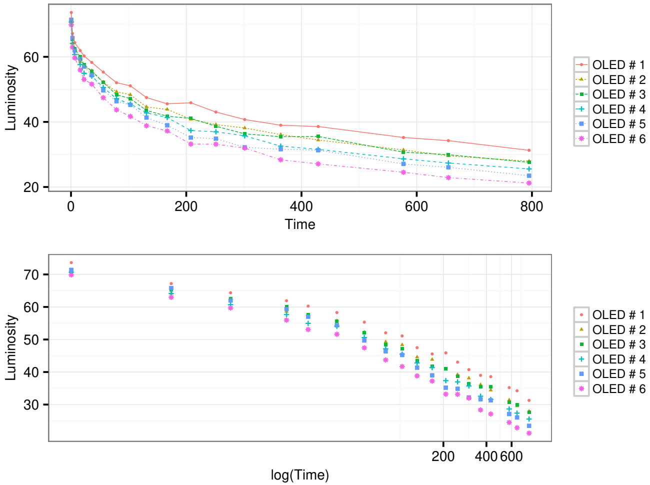


Figure 3: Degradation paths of OLEDs: luminosity against time (top) and luminosity against (log) transformed inspection time (bottom).

Related Literature

- Tseng et al. (1995) to analyze the two-phase degradation data tend to delete early degradation measurements.
- Bae and Kvam (2006) introduces a change-point regression model to fit degradation paths.
- A bi-exponential model with random-coefficients is proposed in Bae et al. (2008) and compared with a exponential model.
- Bae et al. (2015) adopt a Bayesian approach to model the two-phase degradation by using a change-point regression model under the continuity constraint.
- With the prior information taken into account, the bi-exponential model is reestablished in Yuan et al. (2016) under the Bayesian framework.

Why Wiener Process and Measurement Error?

- From the physical point of view, for many products, the degradation increment in an infinitesimal time interval can be viewed as an additive superposition of a large number of small external effects.
- See Wang (2010) studies Wiener process with random effects for degradation data.
- The objective Bayesian method is developed for the accelerated degradation test based on Wiener process in Guan et al. (2016).
- Ye et al. (2013) incorporate the measurement error in the Wiener process on account of the imperfect inspection.

Motivation

Motivation

- This study is mainly motivated by the two-phase degradation data of the OLEDs with the luminosity (or brightness) as the critical characteristics.
- The OLEDs initially decrease rapidly and after some time points the degradation processes become stable. Thus this inspires us to introduce the change-point in our model to present the time point of the transition between two phases.

Contributions

- We propose a change-point Wiener process with measurement error (CPWPME) through specifying the drift of the Wiener process as a two-phase linear function of time.
- Besides, the variability of the degradation paths for different OLEDs drives us to consider the unit-specific coefficients and change-points in the drift function.

Wiener process

Definition of Wiener process

- $W(t)$: the observed degradation character at t
- $Y(t) = W(0) - W(t)$: degradation value at t
- A well-adopted form of the Wiener process is written as

$$X(t) = m(t) + \sigma\mathcal{B}(t), \tag{1}$$

where $m(t)$ is the drift, σ is the diffusion coefficient, and $\mathcal{B}(t)$ is the standard Brownian motion with properties: i) $\mathcal{B}(0) = 0$; ii) $\mathcal{B}(t), t > 0$, has stationary independent Gaussian increments, i.e. $\Delta\mathcal{B} = \mathcal{B}(t + \Delta t) - \mathcal{B}(t)$ follows a normal distribution $\mathcal{N}(0, \Delta t)$.

Two-phase Wiener degradation process

Drift function of ith unit

The drift function of ith unit, $i = 1, \ldots, n$, where n is the number of units, $m_i(t; \beta_i^H, \beta_i^L, \tau_i)$ is formulated as

$$m_i(t; \beta_i^H, \beta_i^L, \tau_i) = \begin{cases} \beta_i^H t, & \text{if } t \leq \tau_i \\ \beta_i^L (t - \tau_i) + \beta_i^H \tau_i, & \text{if } t > \tau_i, \end{cases} \tag{2}$$

where β_i^H is the higher degradation rate at the early stage, β_i^L is the lower degradation rate at the stable stage, and τ_i is the change-point for the ith individual unit.

Notation

- $\boldsymbol{t}_i \equiv (t_{i,1}, \ldots, t_{i,n_i})$: the ordered inspection time points for the ith unit.
- $\boldsymbol{y}_i \equiv (y_{i,1}, \ldots, y_{i,n_i})$: the corresponding observed degradations of $\boldsymbol{Y}_i \equiv (Y_{i,1}, \ldots, Y_{i,n_i})$.
- n_i: the number of inspection time points.
- n: number of unit.
- $X_{i,j} = X(t_{i,j})$.
- $\Delta y_{i,j} \equiv (y_{i,j+1} - y_{i,j})$: the observed degradation increment of $\Delta Y_{i,j} \equiv (Y_{i,j+1} - Y_{i,j})$ on the time interval $(t_{i,j}, t_{i,j})$.
- $\Delta t_{i,j} = t_{i,j+1} - t_{i,j}$.

Proposed model: CPWPME

$$Y_{i,j} = X_{i,j} + \epsilon_{i,j}, \tag{3}$$

where $\epsilon_{i,j}$ is the measurement error and follows $\mathcal{N}(0, \gamma^2)$.

Statistical Properties of $\Delta Y_{i,j}$

Expectation

$$\Delta m_{i,j} = \begin{cases} \beta_i^H \Delta t_{i,j}, & \text{if } \tau_i \geq t_{i,j+1}, \\ \beta_i^H(\tau_i - t_{i,j}) + \beta_i^L(t_{i,j+1} - \tau_i), & \text{if } t_{i,j} \leq \tau_i < t_{i,j+1}, \\ \beta_i^L \Delta t_{i,j}, & \text{if } \tau_i < t_{i,j}, \end{cases}$$

Covariance between $\Delta Y_{i,g}$ and $\Delta Y_{i,k}$

$$\text{cov}(\Delta Y_{i,g}, \Delta Y_{i,k}) = \begin{cases} \sigma^2 \Delta t_{i,1} + \gamma^2, & \text{if } k = g = 1, \\ \sigma^2 \Delta t_{i,k} + 2\gamma^2, & \text{if } k = g > 1, \\ -\gamma^2, & \text{if } k = g + 1 \text{ or } g = k + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $k, g = 1, \ldots, n_i - 1$.

Joint probability density function (PDF) of $\Delta \boldsymbol{Y}_i$

- $\Delta \boldsymbol{m}_i \equiv (\Delta m_{i,1}, \ldots, \Delta m_{i,n_i-1})$: the mean vector.
- Σ_i: the covariance matrix with the (k, g)th element given by $\text{cov}(\Delta Y_{i,g}, \Delta Y_{i,k})$ for the ith degradation increment vector.
- $\Delta \boldsymbol{Y}_i \equiv (\Delta Y_{i,1} \ldots, \Delta Y_{i,n_i-1})$.

Joint PDF of $\Delta \boldsymbol{Y}_i$

$$f_{\Delta \boldsymbol{Y}_i}(\Delta \boldsymbol{y}_i) = (2\pi)^{-\frac{n_i-1}{2}} |\Sigma_i|^{-\frac{1}{2}} \exp\left[-\frac{(\Delta \boldsymbol{y}_i - \Delta \boldsymbol{m}_i)^\intercal \Sigma_i^{-1} (\Delta \boldsymbol{y}_i - \Delta \boldsymbol{m}_i)}{2}\right],$$

where $\Delta \boldsymbol{y}_i \equiv \{\Delta y_{i,1} \ldots, \Delta y_{i,n_i-1}\}$ is the ith observed degradation increment vector.

Likelihood Function

- $\boldsymbol{\beta}^H \equiv (\beta_1^H, \ldots, \beta_n^H)$: the higher degradation rate parameter vector.
- $\boldsymbol{\beta}^L \equiv (\beta_1^L, \ldots, \beta_n^L)$: the lower degradation rate parameter vector.
- $\boldsymbol{\tau} \equiv (\tau_1, \ldots, \tau_n)$: the change-point parameter vector.
- $(\boldsymbol{\beta}^H, \boldsymbol{\beta}^L, \boldsymbol{\tau}, \sigma^2, \gamma^2)$: a set of all the parameters in the CPWPME model.

Likelihood function of $(\boldsymbol{\beta}^H, \boldsymbol{\beta}^L, \boldsymbol{\tau}, \sigma^2, \gamma^2)$

$$
\begin{aligned}
&\mathcal{L}(\boldsymbol{\beta}^H, \boldsymbol{\beta}^L, \boldsymbol{\tau}, \sigma^2, \gamma^2) \\
&= \prod_{i=1}^{n} (2\pi)^{-\frac{n_i-1}{2}} |\Sigma_i|^{-\frac{1}{2}} \exp\left[-\frac{(\Delta \boldsymbol{y}_i - \Delta \boldsymbol{m}_i)^\intercal \Sigma_i^{-1} (\Delta \boldsymbol{y}_i - \Delta \boldsymbol{m}_i)}{2}\right]. \qquad (4)
\end{aligned}
$$

Prior Specification

1. A truncated trivariate normal distribution is assigned for $\boldsymbol{\eta}_i$, for $i = 1, \dots, n$, i.e. $\boldsymbol{\eta}_i \equiv (\beta_i^H, \beta_i^L, \tau_i) \sim \mathcal{MVN}(\boldsymbol{\omega}, \Omega)\mathcal{I}_{\{\beta_i^H > 0,\ \beta_i^L > 0,\ \tau_i > 0,\}}$, where $\boldsymbol{\omega}$ is the mean vector and Ω is the covariance matrix, and $\mathcal{I}_{\{\beta_i^H < 0,\ \beta_i^L < 0,\ \tau_i < 0,\}}$ is the indicator function.
2. The conjugate prior for $\boldsymbol{\omega}$ is also a trivariate normal distribution $\mathcal{MVN}(\boldsymbol{\kappa}, \Psi)$. Let the mean vector $\boldsymbol{\kappa} = \mathbf{0}_3$ and the covariance matrix $\Psi = 10^{-6}\boldsymbol{I}_3$, where $\mathbf{0}_3$ is a three dimensional zero vector and $\boldsymbol{I}_3$ is a 3×3 identity matrix.
3. Decompose the Ω as $\Omega = \Theta Q \Theta$, where $\Theta = \text{diag}\{\theta_1, \theta_2, \theta_3\}$. Assign the inverse-Wishart distribution $\mathcal{IW}(\rho, \boldsymbol{S})$ for Q. Specify the Gamma distribution $\mathcal{G}(a_\theta, b_\theta)$ as the prior distribution of θ_k for $k = 1, 2, 3$. Let $\rho = 4$, $S = \boldsymbol{I}_3$, and $a_\theta = 0.0001$, $b_\theta = 0.0001$.
4. The inverse Gamma distributions $\mathcal{IG}(a_\sigma, b_\sigma)$ and $\mathcal{IG}(a_\gamma, b_\gamma)$ are assigned for σ^2 and γ^2 respectively. Let $a_\sigma = b_\sigma = a_\gamma = b_\gamma = 0.001$.

Posterior Inference

- Define all the parameters vector as $\boldsymbol{\theta} \equiv (\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_n, \sigma^2, \gamma^2, Q, \theta_1, \theta_2, \theta_3)$.

Joint posterior distribution of $\boldsymbol{\theta}$

$$\begin{aligned} \pi(\boldsymbol{\theta}|y) \propto & \mathcal{L}(\boldsymbol{\beta}^H, \boldsymbol{\beta}^L, \boldsymbol{\tau}, \sigma^2, \gamma^2) \left[\prod_{i=1}^{n} \pi(\boldsymbol{\eta}_i|\boldsymbol{\omega}, \Omega)\right] \pi(\boldsymbol{\omega}|\boldsymbol{\kappa}, \Psi)\pi(Q|\rho, S) \\ & \times \left[\prod_{k=1}^{3} \pi(\theta_k|a_k, b_k)\right] \pi(\sigma^2|a_\sigma, b_\sigma)\pi(\gamma^2|a_\gamma, b_\gamma) \end{aligned} \tag{5}$$

Failure-time

- The OLED devices are regarded to have failed if their luminosity fall below 50% of their initial luminosity.
- Define the 50% of six OLEDs' initial luminosity as a vector $(\mathcal{F}_1, \ldots, \mathcal{F}_6)$.
- Failure-time of the ith testing unit is defined as $T_i = \inf\{t | Y(t) \leq \mathcal{F}_i\}$, where $\mathcal{F}_i$ is the failure threshold of ith device.

Cumulative distribution function (CDF) of the failure-time

$$F_{T_i}(t) = \begin{cases} F_{IG}\left(t; \frac{\mathcal{F}_i}{\beta_i^H}, \frac{\mathcal{F}_i^2}{\sigma^2}\right), & \text{if } t \leq \tau_i, \\ F_{IG}\left(t; \frac{\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i}{\beta_i^L}, \frac{(\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i)^2}{\sigma^2}\right), & \text{if } t > \tau_i, \end{cases} \tag{6}$$

for $i = 1, \ldots, N$. Here, $F_{IG}(x; \mu, \lambda)$ denotes an inverse Gaussian (IG) distribution with mean vector μ and shape parameter λ.

Mean time to failure (MTTF)

MTTF of each OLED device

$$\mathcal{E}[T_i] = \frac{\mathcal{F}_i}{\beta_i^H}\left[1 - F_{IG}\left(\frac{\mathcal{F}_i^2}{\tau_i {\beta_i^H}^2}; \frac{\mathcal{F}_i}{\beta_i^H}, \frac{\mathcal{F}_i^2}{\sigma^2}\right)\right] + \frac{\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i}{\beta_i^L}$$
$$\times F_{IG}\left(\frac{[\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i]^2}{\tau_i {\beta_i^L}^2}; \frac{\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i}{\beta_i^L}, \frac{[\mathcal{F}_i - (\beta_i^H - \beta_i^L)\tau_i]^2}{\sigma^2}\right),$$

for $i = 1, \ldots, N$.

Simulation Study

1. The CPWPME data are randomly generated under the following three different setup for the number of units and the number of inspection time points, i.e

   Scenario I: $n = 5$, $n_i = 16$;

   Scenario II: $n = 5$, $n_i = 21$;

   Scenario III: $n = 10$, $n_i = 21$.

2. The inspection time points are chosen from 0 to 18 with identical time intervals under each scenario.

Simulation Study

- Degradation increments are generated by sampling from a multivariate normal distribution.
- For parameter estimation, we apply the hierarchical Bayesian method. We initiate with a 15,000-iteration burn-in period, using the *coda* package in *R* for convergence diagnostics via trace, ergodic mean, autocorrelation plots, and the Gelman-Rubin ratio.
- An additional 10,000 iterations generate posterior samples for parameter inference, estimating each parameter by the posterior mean.
- This simulation and estimation cycle is repeated 500 times for each data configuration.

Table 1: Parameter estimation results from the HB approach and the ML method for scenario I.

Stat.	β_1^H	β_2^H	β_3^H	β_4^H	β_5^H	β_1^L	β_2^L	β_3^L	β_4^L	β_5^L
True	6.720	7.082	6.626	7.713	7.147	1.741	2.154	2.233	2.182	1.903
Bias	0.142	0.017	0.177	-0.259	-0.030	0.223	-0.061	-0.096	-0.122	0.128
SE	0.318	0.312	0.341	0.398	0.325	0.465	0.430	0.416	0.422	0.480
RMSE	0.348	0.312	0.384	0.474	0.326	0.516	0.434	0.427	0.439	0.496
CP	0.928	0.990	0.910	0.876	0.960	0.960	0.984	0.968	0.960	0.966
Stat.	τ_1	τ_2	τ_3	τ_4	τ_5	$\omega[1]$	$\omega[2]$	$\omega[3]$	σ^2	γ^2
True	12.828	12.214	11.660	10.787	12.616	7.000	2.000	12.000	2.000	1.000
Bias	-0.284	-0.085	0.074	0.291	-0.209	0.067	0.057	-0.021	0.048	0.185
SE	0.490	0.413	0.523	0.405	0.463	0.202	0.320	0.257	0.704	0.487
RMSE	0.566	0.422	0.527	0.498	0.508	0.213	0.324	0.257	0.705	0.520
CP	0.924	0.966	0.972	0.940	0.940	0.994	0.996	1.000	0.974	0.970

Table 2: Parameter estimation results from the HB approach and the ML method for scenario II.

Stat.	β_1^H	β_2^H	β_3^H	β_4^H	β_5^H	β_1^L	β_2^L	β_3^L	β_4^L	β_5^L
True	6.720	7.082	6.626	7.713	7.147	1.741	2.154	2.233	2.182	1.903
Bias	0.149	-0.007	0.193	-0.297	-0.024	0.239	-0.034	-0.120	-0.136	0.119
SE	0.320	0.294	0.325	0.373	0.276	0.463	0.431	0.410	0.417	0.412
RMSE	0.353	0.294	0.378	0.477	0.276	0.521	0.432	0.427	0.438	0.429
CP	0.934	0.972	0.924	0.884	0.986	0.946	0.974	0.972	0.974	0.986
Stat.	τ_1	τ_2	τ_3	τ_4	τ_5	$\omega[1]$	$\omega[2]$	$\omega[3]$	σ^2	γ^2
True	12.828	12.214	11.660	10.787	12.616	7.000	2.000	12.000	2.000	1.000
Bias	-0.292	-0.052	0.069	0.331	-0.155	0.060	0.057	0.002	0.159	0.085
SE	0.432	0.403	0.448	0.450	0.347	0.191	0.303	0.221	0.717	0.393
RMSE	0.521	0.406	0.453	0.559	0.379	0.200	0.308	0.221	0.733	0.402
CP	0.930	0.974	0.970	0.918	0.970	1.000	1.000	0.998	0.944	0.960

Table 3: Parameter estimation results for scenario III.

Stat.	β_1^H	β_2^H	β_3^H	β_4^H	β_5^H	β_6^H	β_7^H	β_8^H	β_9^H	β_{10}^H
True	6.720	7.082	6.626	7.713	7.147	6.633	7.218	7.330	7.257	6.863
Bias	0.179	-0.022	0.259	-0.362	-0.064	0.249	-0.094	-0.152	-0.113	0.109
SE	0.255	0.227	0.248	0.308	0.219	0.245	0.233	0.251	0.246	0.226
RMSE	0.311	0.228	0.358	0.475	0.228	0.349	0.251	0.294	0.270	0.250
CP	0.908	0.978	0.894	0.814	0.980	0.890	0.978	0.946	0.966	0.976
Stat.	β_1^L	β_2^L	β_3^L	β_4^L	β_5^L	β_6^L	β_7^L	β_8^L	β_9^L	β_{10}^L
True	2.478	2.123	1.804	1.300	2.356	1.986	1.995	2.298	2.260	2.188
Bias	-0.174	0.031	0.195	0.347	-0.108	0.077	0.052	-0.162	-0.125	-0.038
SE	0.378	0.360	0.331	0.433	0.331	0.322	0.303	0.343	0.349	0.326
RMSE	0.416	0.361	0.384	0.555	0.348	0.331	0.307	0.379	0.371	0.328
CP	0.952	0.984	0.960	0.878	0.990	0.986	0.990	0.972	0.976	0.990
Stat.	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
True	12.503	12.428	12.041	10.910	12.339	11.969	11.915	11.194	11.738	12.229
Bias	-0.220	-0.199	-0.129	0.272	-0.128	-0.078	-0.012	0.329	0.093	-0.141
SE	0.369	0.311	0.331	0.349	0.331	0.327	0.292	0.358	0.286	0.345
RMSE	0.430	0.369	0.355	0.442	0.355	0.336	0.292	0.486	0.301	0.372
CP	0.930	0.946	0.970	0.894	0.962	0.968	0.992	0.858	0.980	0.960
Stat.	$\omega[1]$	$\omega[2]$	$\omega[3]$	σ^2	γ^2					
True	7.000	2.000	12.000	2.000	1.000					
Bias	0.058	0.088	-0.094	0.063	0.041					
SE	0.135	0.206	0.159	0.517	0.290					
RMSE	0.147	0.224	0.185	0.520	0.293					
CP	0.992	0.986	0.996	0.952	0.956					

Simulation Study

Key points

1. Our model and method despite the small sample size and large number of parameters.
2. Results from Scenario II show no significant reduction in absolute bias with increased n_i, but SE and RMSE decrease notably compared to Scenario I, achieving better nominal coverage probability.
3. For Scenarios II and III, increases in n do not substantially reduce absolute bias due to a corresponding rise in unknown parameters, yet the hierarchical Bayesian method remains effective even with small sample sizes.

OLED data analysis

- The OLED degradation data was modeled using the CPWPME approach, with parameter estimation conducted via hierarchical methods.
- The Markov chains were initiated with a 20,000 iteration burn-in period, followed by an additional 30,000 iterations to obtain posterior samples for inference.
- Estimation results for the CPWPME model are summarized in Table 4. The estimated posterior means for ω and the covariance matrix Ω are given by:

$$\hat{\boldsymbol{\omega}} = (3.76, 9.74, 4.36)$$

$$\hat{\Omega} = \begin{pmatrix} 0.16350 & 0.00353 & -0.00368 \\ 0.00353 & 0.22370 & -0.00108 \\ -0.00368 & -0.00108 & 0.09092 \end{pmatrix}$$

Table 4: Parameter estimation based on the CPWP model.

OLED	β^H				β^L			
	Est.	SE	2.5%	97.5%	Est.	SE	2.5%	97.5%
#1	3.665	0.224	3.204	4.100	9.800	0.302	9.217	10.440
#2	3.653	0.230	3.177	4.099	9.557	0.331	8.821	10.120
#3	3.697	0.222	3.243	4.143	9.819	0.294	9.271	10.460
#4	3.806	0.215	3.404	4.261	9.635	0.295	8.992	10.170
#5	3.775	0.230	3.323	4.250	9.808	0.290	9.251	10.420
#6	3.932	0.256	3.503	4.488	9.802	0.285	9.234	10.400

OLED	τ			
	Est.	SE	2.5%	97.5%
#1	4.475	0.131	4.210	4.749
#2	4.482	0.144	4.218	4.802
#3	4.364	0.124	4.101	4.586
#4	4.425	0.121	4.193	4.673
#5	4.165	0.152	3.872	4.447
#6	4.266	0.129	4.013	4.506

Models comparison

Benchmark models

- **CPWP**: The CPWP model is similar to our CPWPME model but omits measurement error.
- **TPLCP**: $y_{i,j} = \begin{cases} \zeta_i t_{i,j} - \kappa_i t_{i,j} + \epsilon_{i,j}, & j = 1, \ldots, \gamma_i \\ \zeta_i t_{i,j} - \kappa_i \varsigma_i + \epsilon_{i,j}, & j = \gamma_i + 1, \ldots, n_i \end{cases}$ for the ith item data, where $y_{i,j}$ is the jth observation measured at time $t_{i,j}$, and $\varsigma_i \in [t_{\gamma_i}, t_{\gamma_i+1})$ is the change-point of ith item. The error $\epsilon_{i,j}$ are assumed to be $i.i.d.$ $\mathcal{N}(0, \upsilon^2)$.
- **BE**: $y_{i,j} = \phi_i \exp(-(\gamma_i + \Delta\gamma_i)t_{ij}) + (1 - \phi_i) \exp(-\gamma_i t_{i,j}) + \epsilon_{i,j}$ where $i = 1, \ldots, I, j = 1, \ldots, n_i$, and error $\epsilon_{i,j}$ are assumed to be $i.i.d$ $\mathcal{N}(0, \omega^2)$. Denote $\boldsymbol{\phi} \equiv (\phi_1, \ldots, \phi_I)^\intercal$, $\boldsymbol{\gamma} \equiv (\gamma_1, \ldots, \gamma_I)^\intercal$, and $\Delta\boldsymbol{\gamma} \equiv (\Delta\gamma_1, \ldots, \Delta\gamma_I)^\intercal$.

Table 5: Parameter estimation of defferent benchmark models.

OLED	CPWP Model			TPLCP Model			BE Model		
	β^H	β^L	τ	ζ	κ	ς	ϕ	γ	$\Delta\gamma$
#1	3.799	9.269	4.373	-8.947	-5.440	4.506	0.647	4.741	-4.681
#2	3.797	9.300	4.369	-9.144	-5.640	4.486	0.647	4.741	-4.680
#3	3.815	9.305	4.310	-9.088	-5.560	4.256	0.634	4.742	-4.679
#4	3.849	9.334	4.364	-9.445	-5.555	4.439	0.622	4.743	-4.678
#5	3.850	9.429	4.210	-9.670	-5.936	4.025	0.609	4.745	-4.676
#6	3.903	9.407	4.287	-9.819	-5.654	4.215	0.596	4.745	-4.676

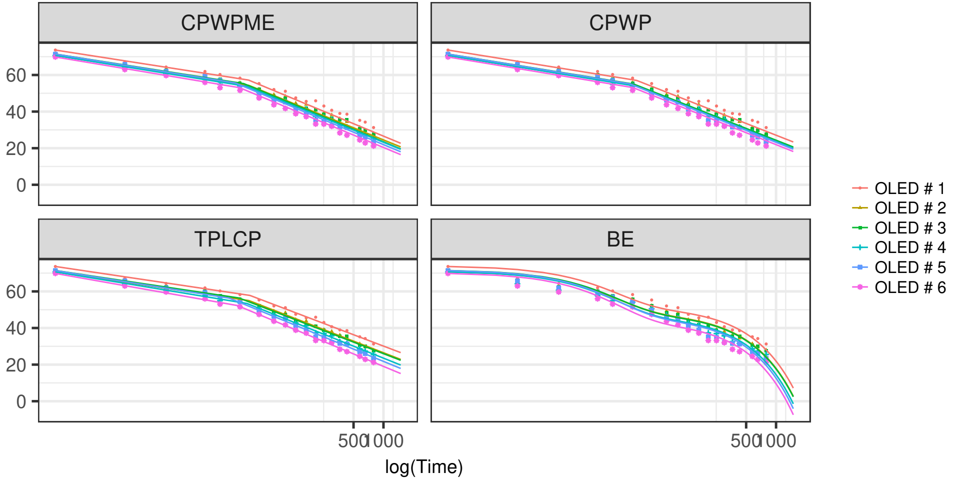


Figure 4: The posterior degradation path fits; luminosity vs. log(time) for each OLED data.

Mean square prediction error (MSPE)

$$\text{MSPE} = \sum_{j=1}^{n_i} (y_{i,j} - \hat{y}_{i,j})^2 , \tag{7}$$

where $\boldsymbol{y}_i = \{y_{i,1}, \dots, y_{i,n_i}\}, i = 7$, is the degradation data of the 7th unit and $\hat{\boldsymbol{y}}_i = \{\hat{y}_{i,1}, \dots, \hat{y}_{i,n_i}\}$ is the corresponding prediction value.

Table 6: MSPE for the 7th OLED degradation path.

Model	CPWPME	CPWP	TPLCP	BE
MSPE	360.04	406.04	436.29	678.53

- The CPWPME model's MSPE is much smaller than that of three other models, indicating its superiority.

MTTF

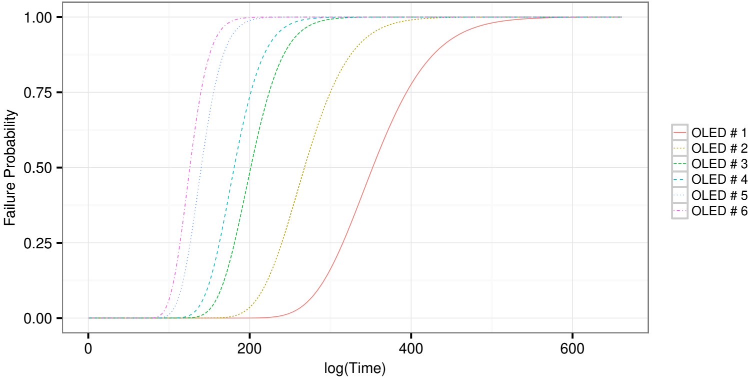


Figure 5: Posterior distribution of the failure-time for each OLED.

- The MTTF estimates for each unit are $(352.28, 267.84, 201.25, 180.67, 139.88, 125.77)$.

Outline

1. Introduction
2. Two-phase degradation model
   - Wiener model
   - Inverse Gaussian model
3. Multivariate degradation model
4. Conclusion

Motivated example: lithium batteries

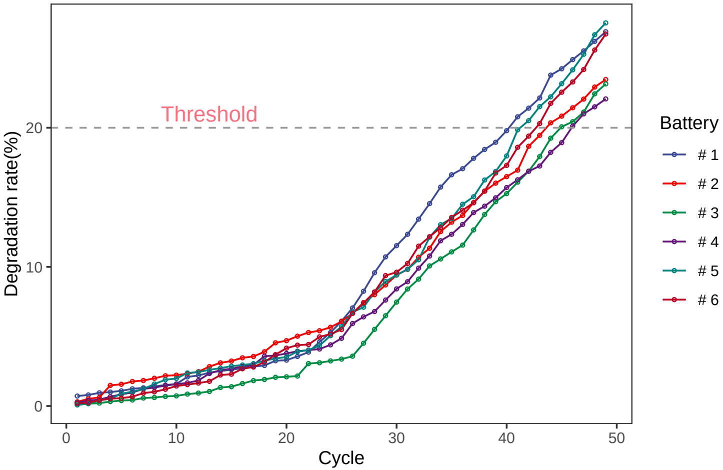
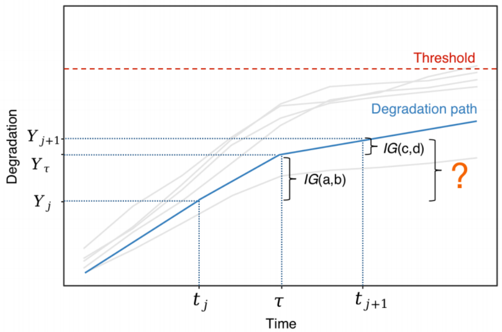


Figure 6: Capacity degradation data of lithium batteries.

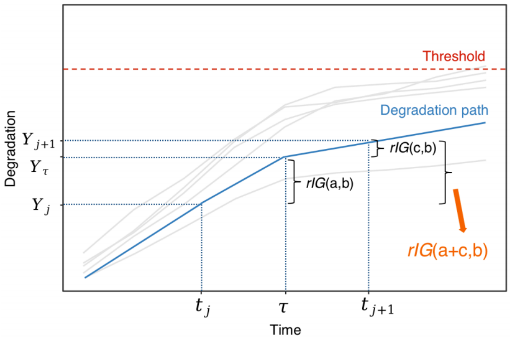
Related Literature

Two-phase degradation modeling

1. Wiener process: Wang et al. (2018a, 2018b), Zhang et al. (2019), Lin et al. (2021), Ma et al. (2023), etc.
2. Gamma process: Ling et al. (2019), Lin et al. (2021).
3. IG process: Duan and Wang (2017).
   - Limitations of Duan and Wang (2017):
     (i) Constraints on locations of change points;
     (ii) Insufficient considerations for deriving the lifetime distribution;
     (iii) Neglecting the uncertainty in estimation.



(a) IG process



(b) Re-parameterized IG process

Contributions

(i) A novel two-phase reparameterized IG (rIG) degradation model with distinct change points and model parameters for each individual system;

(ii) Derive the distribution of failure time and remaining useful life (RUL), and propose an adaptive replacement policy;

(iii) Employ bootstrap and Bayesian approach to generate interval estimates for the parameters.

Reparameterized IG distribution

Connection to IG distribution

The rIG distribution $rIG(\delta, \gamma)$ relates to the traditional IG distribution $IG(a, b)$ as $a = \delta/\gamma$ and $b = \delta^2$.

Moment generating function (MGF)

$$M_Y(t) = E(e^{ty}) = e^{\delta\gamma\left(1-\sqrt{1-\frac{2t}{\gamma^2}}\right)}. \quad (8)$$

Additive property

If $Y_1 \sim rIG(\delta_1, \gamma), Y_2 \sim rIG(\delta_2, \gamma)$, then $Y_1 + Y_2 \sim rIG(\delta_1 + \delta_2, \gamma)$.

PDF

If a random variable Y follows rIG distribution, then its PDF is

$$f_{rIG}(y|\delta,\gamma) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} y^{-3/2} e^{-\left(\delta^2 y^{-1} + \gamma^2 y\right)/2}, \; y > 0, \; \delta > 0, \; \gamma > 0. \tag{9}$$

CDF

$$F_{rIG}(y|\delta,\gamma) = \Phi\left[\sqrt{y}\gamma - \frac{\delta}{\sqrt{y}}\right] + e^{2\delta\gamma}\Phi\left[-\sqrt{y}\gamma - \frac{\delta}{\sqrt{y}}\right], \tag{10}$$

where $\Phi(\cdot)$ is the CDF of the standard normal distribution.

rIG process

Definition of rIG process

rIG process $\{Z(t), t \geq 0\}$ satisfies the following properties:

(i) $Z(0) = 0$ with probability one;

(ii) $Z(t)$ has independent increments. Specifically, $Z(t_2) - Z(t_1)$ and $Z(s_2) - Z(s_1)$ are independent for all $t_2 > t_1 \geq s_2 > s_1 \geq 0$;

(iii) For all $t > s \geq 0$, $Z(t) - Z(s)$ follows the rIG distribution $rIG(\delta(\Lambda(t) - \Lambda(s)), \gamma)$, where $\Lambda(t)$ is a monotone increasing function with $\Lambda(0) = 0$, δ and γ are unknown parameters.

- Denoted as $r\mathcal{IG}(\delta\Lambda(t), \gamma)$.
- The mean and variance of $\{Z(t), t \geq 0\}$, which are $\delta\Lambda(t)/\gamma$ and $\delta\Lambda(t)/\gamma^3$, respectively.

Two-phase rIG degradation model

Two-phase rIG degradation model

Suppose a system's performance characteristic degrades in two distinct phases, separated by a single change point.

$$
\begin{aligned}
& Y(t)|\tau \sim r\mathcal{IG}\left(m(t;\delta_1,\delta_2,\tau),\gamma\right), \ \tau \sim N\left(\mu_\tau,\sigma_\tau^2\right), \\
& m(t;\delta_1,\delta_2,\tau) = \begin{cases} \delta_1 t, & t \le \tau, \\ \delta_2\left(t-\tau\right)+\delta_1\tau, & t > \tau, \end{cases}
\end{aligned} \tag{11}
$$

where δ_1 and δ_2 are the drift parameters for $t \le \tau$ and $t > \tau$, respectively.

Failure-time

Let $T = \inf\{t \mid Y(t) \geq \mathcal{D}\}$, and $Y(t) = \begin{cases} Y_1(t), & t \leq \tau, \\ Y_1(\tau) + Y_2(t-\tau), & t > \tau. \end{cases}$

Conditional reliability function of T

- $0 \leq t \leq \tau$

$$\bar{F}_1(t \mid \tau) = P(T > t \mid \tau \geq t) = P(Y_1(t) < \mathcal{D} \mid \tau \geq t) = F_{r\mathcal{IG}}(\mathcal{D} | \delta_1 t, \gamma). \tag{12}$$

- $t > \tau$

$$\begin{aligned} \bar{F}_2(t \mid \tau) =& P\left(Y(t) < \mathcal{D} \mid \tau < t\right) = P\left(Y_1(\tau) + Y_2(t-\tau) < \mathcal{D} \mid \tau < t\right) \\ =& \int_0^{\mathcal{D}} F_{r\mathcal{IG}}(\mathcal{D} - y_\tau | \delta_2(t-\tau), \gamma) f_1(y_\tau \mid \tau) \mathrm{d} y_\tau, \end{aligned} \tag{13}$$

where y_τ represents the degradation value at τ, and $f_1(y_\tau \mid \tau)$ is the PDF of y_τ.

Failure-time

Unconditional reliability function of T

$$\begin{aligned} R(t) &= P\left(Y(t) < \mathcal{D}, \tau \geq t\right) + P\left(Y(t) < \mathcal{D}, 0 < \tau < t\right) \\ &= \bar{F}_1\left(t \mid \tau\right) \bar{G}_\tau(t) + \int_0^t g_\tau(\tau|\mu_\tau, \sigma_\tau^2) \bar{F}_2\left(t \mid \tau\right) d\tau, \end{aligned} \tag{14}$$

where $\bar{G}_\tau(t)$ is the survival function of random variable τ.

MTTF

$$\text{MTTF} = E(T) = \int_0^\infty R(t) dt. \tag{15}$$

RUL

Let $S_t = \inf\{x; Y(t+x) \geq \mathcal{D} \mid Y(t) < \mathcal{D}\}$.

Conditional reliability function of S_t

(i) When $x + t \leq \tau$:

$$\bar{F}_{S_t,1}(x \mid \tau) = F_{r\mathcal{IG}}(\mathcal{D} - Y(t)|\delta_1 x, \gamma). \tag{16}$$

(ii) When $t < \tau < x + t$:

$$\begin{aligned}\bar{F}_{S_t,2}(x \mid \tau) &= P(Y(t+x) < \mathcal{D} \mid Y(t) \leq \mathcal{D}) \\ &= \int_0^{\mathcal{D}} F_{r\mathcal{IG}}(\mathcal{D} - y_\tau|\delta_2(t+x-\tau), \gamma) f_1(y_\tau \mid \tau)\mathrm{d}y_\tau.\end{aligned} \tag{17}$$

(iii) When $\tau \leq t$:

$$\bar{F}_{S_t,3}(x \mid \tau) = F_{r\mathcal{IG}}(\mathcal{D} - Y(t)|\delta_2 x, \gamma). \tag{18}$$

RUL

Unconditional reliability function of S_t

$$\begin{aligned} R_{S_t}(x) =& P(Y(t+x) < \mathcal{D}, t < x+t \leq \tau) \\ & + P(Y(t+x) < \mathcal{D}, t \leq \tau < x+t) + P(Y(t+x) < \mathcal{D}, t > \tau) \\ =& \bar{F}_{S_t,1}(x \mid \tau)\,\bar{G}_\tau(x+t) + \int_t^{x+t} g_\tau(\tau|\mu_\tau, \sigma_\tau^2)\bar{F}_{S_t,2}(x \mid \tau)\,d\tau \\ & + \int_0^t g_\tau(\tau)\bar{F}_{S_t,3}(x \mid \tau)\,d\tau. \end{aligned} \tag{19}$$

Mean of RUL at time t

$$\text{MRL} = E(S_t) = \int_0^\infty R_{S_t}(x)dx. \tag{20}$$

Data

- I systems under inspection in a degradation test.
- Deterioration pattern follows the two-phase rIG degradation model.
- $Y_{i,j}$ is the observed degradation value at the measurement time $t_{i,j}$, $i = 1\ldots, I,\ j = 1, \ldots, n_i$, and $0 < t_{i,1} < \ldots < t_{i,n_i}$.
- Let $\Delta y_{i,j} = Y_{i,j} - Y_{i,j-1}$, $Y_{i,0} = 0$.
- Denote $\boldsymbol{\Delta Y}_i = (\Delta y_{i,1}, \ldots, \Delta y_{i,n_i})^\top$, $\boldsymbol{\Delta Y} = \left(\boldsymbol{\Delta Y}_1^\top, \cdots, \boldsymbol{\Delta Y}_I^\top\right)^\top$.

Conditional PDF of $\Delta y_{i,j}$

$$\Delta y_{i,j} \sim rIG\left(\Delta m_{i,j}^{(k)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right), \gamma\right),$$

$$\Delta m_{i,j}^{(k)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right) = \begin{cases} \delta_{1,i}\Delta t_{i,j} & k=1, \\ \left(\delta_{1,i}-\delta_{2,i}\right)\tau_i + \delta_{2,i}t_{i,j} - \delta_{1,i}t_{i,j-1}, & k=2, \\ \delta_{2,i}\Delta t_{i,j}, & k=3, \end{cases}$$

$\Delta t_{i,j} = t_{i,j} - t_{i,j-1}$ and $t_{i,0} = 0$, $i = 1\ldots, I,\ j = 1, \ldots, n_i$.

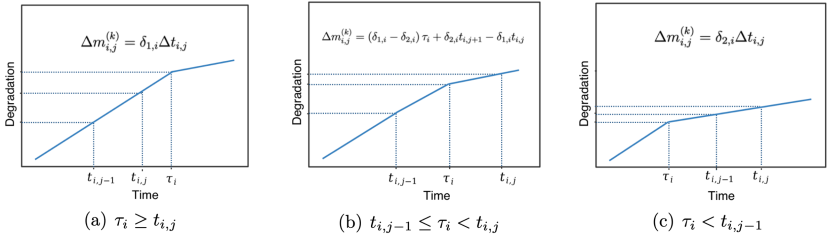


Figure 7: Three scenarios for change points and inspection time.

Conditional PDF of $\Delta y_{i,j}$

Let $\lambda_{i,j}^{(1)} = \mathcal{I}\left(\tau_i \geq t_{i,j}\right), \lambda_{i,j}^{(2)} = \mathcal{I}\left(t_{i,j-1} \leq \tau_i < t_{i,j}\right), \lambda_{i,j}^{(3)} = \mathcal{I}\left(\tau_i < t_{i,j-1}\right)$.

$$\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right) = \Delta m_{i,j}^{(1)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(1)}} \times \Delta m_{i,j}^{(2)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(2)}} \times \Delta m_{i,j}^{(3)}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)^{\lambda_{i,j}^{(3)}}.$$

$$\begin{aligned} f_{i,j}\left(\Delta y_{i,j} \mid \delta_{1,i}, \delta_{2,i}, \tau_i, \gamma\right) = & \frac{\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)}{\sqrt{2\pi}} \exp\left\{\gamma \Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)\right\} \Delta y_{i,j}^{-3/2} \\ & \times \exp\left\{-\frac{\left[\Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)\right]^2 \Delta y_{i,j}^{-1} + \gamma^2 \Delta y_{i,j}}{2}\right\}. \end{aligned}$$

Likelihood

- Let $\boldsymbol{\delta}_1 = (\delta_{1,1}, \ldots, \delta_{1,I})^\top$, $\boldsymbol{\delta}_2 = (\delta_{2,1}, \ldots, \delta_{2,I})^\top$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_I)^\top$.
- Denote $\boldsymbol{\eta} = \left(\boldsymbol{\delta}_1^\top, \boldsymbol{\delta}_2^\top, \gamma\right)^\top$, $\boldsymbol{\theta}_\tau = \left(\mu_\tau, \sigma_\tau^2\right)^\top$ and $\boldsymbol{\vartheta} = \left(\boldsymbol{\theta}_\tau^\top, \boldsymbol{\eta}^\top\right)^\top$.
- Given the observed data $\boldsymbol{\Delta Y}$, the likelihood function is

$$L_{obs}(\boldsymbol{\Delta Y}|\boldsymbol{\vartheta}) = \prod_{i=1}^{I} \int_{-\infty}^{\infty} \prod_{j=1}^{n_i} f_{i,j}\left(\Delta y_{i,j} \mid \delta_{1,i}, \delta_{2,i}, \tau_i, \gamma\right) g_\tau(\tau_i|\boldsymbol{\theta}_\tau)\mathrm{d}\tau_i. \tag{21}$$

Remark: Obtain a closed-form solution for the ML estimates of $\boldsymbol{\vartheta}$ is not feasible.

EM Algorithm

Log-likelihood function for the complete data

$$l_c(\boldsymbol{\Delta Y}, \boldsymbol{\tau} | \boldsymbol{\vartheta}) = \sum_{i=1}^{I} l_i\left(\boldsymbol{\theta}_\tau\right) + \sum_{i=1}^{I} \sum_{j=1}^{n_i} l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}), \tag{22}$$

$$l_i\left(\boldsymbol{\theta}_\tau\right) = \log g_\tau\left(\tau_i \mid \boldsymbol{\theta}_\tau\right) = -\log \sqrt{2\pi}\sigma_\tau - \frac{\left(\tau_i - \mu_\tau\right)^2}{2\sigma_\tau^2},$$

$$\begin{aligned} l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) &= \log f_{i,j}\left(\Delta y_{i,j} \mid \boldsymbol{\eta}, \boldsymbol{\tau}\right) \\ &= -\log \sqrt{2\pi} + \log \Delta m_{i,j} + \gamma \Delta m_{i,j} - \frac{3}{2} \log \Delta y_{i,j} - \frac{\Delta m_{i,j}^2}{2\Delta y_{i,j}} - \frac{\gamma^2 \Delta y_{i,j}}{2}, \end{aligned}$$

and $\Delta m_{i,j} = \Delta m_{i,j}\left(\delta_{1,i}, \delta_{2,i}, \tau_i\right)$.

EM Algorithm

- E-step:

$$\begin{aligned} \boldsymbol{Q}_{(s)}(\boldsymbol{\vartheta}) &= E_{\boldsymbol{\vartheta}_{(s)}}\left[l_c(\boldsymbol{\Delta Y}, \boldsymbol{\tau}|\boldsymbol{\vartheta})\right] \\ &= \sum_{i=1}^{I} E_{\boldsymbol{\vartheta}_{(s)}}\left[l_i\left(\boldsymbol{\theta}_{\tau}\right) \mid \boldsymbol{\Delta Y}\right] + \sum_{i=1}^{I}\sum_{j=1}^{n_i} E_{\boldsymbol{\vartheta}_{(s)}}\left[l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) \mid \boldsymbol{\Delta Y}\right], \end{aligned} \tag{23}$$

- M-step:

$$\boldsymbol{\vartheta}_{(s+1)} = \arg\max \boldsymbol{Q}_{(s)}(\boldsymbol{\vartheta}). \tag{24}$$

EM Algorithm

- **Step 1**. Initialize the parameters $\boldsymbol{\vartheta}$ to some random values $\boldsymbol{\vartheta}_{(\mathbf{0})}$, and setting the tolerance error ϵ.
- **Step 2**. Calculate $E_{\boldsymbol{\vartheta}_{(s)}}\left[l_i\left(\boldsymbol{\theta}_\tau\right) \mid \boldsymbol{\Delta y}\right]$ and $E_{\boldsymbol{\vartheta}_{(s)}}\left[l_{i,j}(\boldsymbol{\eta}, \boldsymbol{\tau}) \mid \boldsymbol{\Delta y}\right]$ based on the solution of the s-th iteration $\boldsymbol{\vartheta}_{(s)}$.
- **Step 3**. Calculate the solution of the $(s+1)$-th iteration $\boldsymbol{\vartheta}_{(s+1)}$ by (24).
- **Step 4**. Repeat Steps 2 and 3 until $\left|\boldsymbol{\vartheta}_{(s+1)} - \boldsymbol{\vartheta}_{(s)}\right| < \epsilon$, where $|\cdot|$ is the Euclidean distance.
- **Step 5**. The MLE of $\boldsymbol{\vartheta}$ can be obtained as $\hat{\boldsymbol{\vartheta}} = \boldsymbol{\vartheta}_{(s+1)}$.

Parametric bootstrap method

Algorithm 1: Parametric bootstrap algorithm.

Input: Point estimate $\hat{\boldsymbol{\vartheta}}$.

Output: $\mathcal{B}$ bootstrap estimates $\left\{\hat{\boldsymbol{\vartheta}}_1^*, \ldots, \hat{\boldsymbol{\vartheta}}_{\mathcal{B}}^*\right\}$.

```
for b = 1 to B do
    Generate τ from N(μ̂_τ, σ̂²_τ);
    for i = 1 to I do
        for j = 1 to n_i do
            Generate degradation sample ΔỸ_{i,j} from
              rIG(Δm^{(k)}_{i,j}(δ̂_{1,i}, δ̂_{2,i}, τ̂_i), γ̂), k = 1, 2, 3.
        end
    end
    Obtain ϑ̂*_b based on ΔỸ using the proposed EM algorithm.
end
```

Where: line 2: Generate $\boldsymbol{\tau}$ from $\mathcal{N}(\hat{\mu}_{\tau}, \hat{\sigma}_{\tau}^2)$; line 5: Generate degradation sample $\Delta\tilde{Y}_{i,j}$ from $rIG\left(\Delta m_{i,j}^{(k)}\left(\hat{\delta}_{1,i}, \hat{\delta}_{2,i}, \hat{\tau}_i\right), \hat{\gamma}\right), k = 1, 2, 3.$; line 8: Obtain $\hat{\boldsymbol{\vartheta}}_b^*$ based on $\Delta\tilde{\boldsymbol{Y}}$ using the proposed EM algorithm.

Parametric bootstrap method

After acquiring the bootstrap estimates $\left\{\hat{\boldsymbol{\vartheta}}_1^*, \ldots, \hat{\boldsymbol{\vartheta}}_{\mathcal{B}}^*\right\}$, an approximate $100(1-\alpha)\%$ bootstrap confidence interval for a function of the parameters $h(\boldsymbol{\vartheta})$ is:

$$\left[h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{(\alpha\mathcal{B}/2)}, h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{((1-\alpha/2)\mathcal{B})}\right],$$

where $h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{(b)}$ denotes the b-th statistic among $\left\{h\left(\hat{\boldsymbol{\vartheta}}^*\right)_1, \ldots, h\left(\hat{\boldsymbol{\vartheta}}^*\right)_{\mathcal{B}}\right\}$.

Bayesian analysis

$$Y_i(t|\tau_i) \sim r\mathcal{IG}\left(m(t;\delta_{1,i},\delta_{2,i},\tau_i),\gamma\right),\ \tau_i \sim N\left(\mu_\tau,\sigma_\tau^2\right),\ i=1,\ldots,I,$$

$$m(t;\delta_{1,i},\delta_{2,i},\tau_i) = \begin{cases} \delta_{1,i}t, & t \le \tau_i, \\ \delta_{2,i}\left(t-\tau_i\right)+\delta_{1,i}\tau_i, & t > \tau_i, \end{cases}$$

$$\left(\mu_\tau,\sigma_\tau^2\right) \sim NIGa\left(\beta_\tau,\eta_\tau,\upsilon_\tau,\xi_\tau\right),\gamma \sim N(\omega,\kappa^2),$$

$$\delta_{1,i} \sim N\left(\mu_1,\sigma_1^2\right),\delta_{2,i} \sim N\left(\mu_2,\sigma_2^2\right),$$

$$\left(\mu_1,\sigma_1^2\right) \sim NIGa\left(\beta_1,\eta_1,\upsilon_1,\xi_1\right),\left(\mu_2,\sigma_2^2\right) \sim NIGa\left(\beta_2,\eta_2,\upsilon_2,\xi_2\right),$$

where $NIGa(\cdot)$ denotes the normal-inverse gamma distribution.

Joint posterior distribution of $\boldsymbol{\theta}$

- Let $\boldsymbol{\theta} = \left(\boldsymbol{\vartheta}, \mu_1, \sigma_1^2, \mu_2, \sigma_2^2\right)^\top$ be the parameter vector.
- According to Bayes' theorem, the joint posterior distribution of $\boldsymbol{\theta}$ can be derived as

$$\begin{aligned}\pi(\boldsymbol{\theta} \mid \boldsymbol{\Delta Y}) \propto \pi\left(\mu_\tau, \sigma_\tau^2\right) \pi\left(\mu_1, \sigma_1^2\right) \pi\left(\mu_2, \sigma_2^2\right) \pi\left(\gamma \mid \omega, \kappa\right) \pi\left(\tau \mid \mu_\tau, \sigma_\tau^2\right) \\ \times \pi\left(\boldsymbol{\delta}_1 \mid \mu_1, \sigma_1^2\right) \pi\left(\boldsymbol{\delta}_2 \mid \mu_1, \sigma_1^2\right) f_{\Delta Y}\left(\boldsymbol{\Delta Y} \mid \boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\tau}, \gamma\right).\end{aligned} \tag{25}$$

- Employ the **Gibbs sampling algorithm** to generate posterior samples of the parameters, thereby facilitating Bayesian inference.

Adaptive replacement policy

- $0 = t_{i,0} < t_{i,1} < \cdots < t_{i,j}$ are discrete inspection times.
- $y_{i,j}$ represents the observed degradation value, $y_{i,1:j} = \{y_{i,1}, y_{i,2}, \ldots, y_{i,j}\}$.
- Iteratively update estimations of model parameters and RUL distributions, $f_{S_t}(x|y_{i,1:j})$.

Idea

1. Evaluate **candidate maintenance actions** at each inspection time point.
2. Determine **optimal preparation and maintenance actions** as data continues to be collected.

Policy assumption

- Maintenance is executed perfectly by replacing the system spare parts.
- Failure is detected only by inspections, and the cost of each inspection is c_i.
- An adequate supply of spare parts.
- Maintenance preparation time ϖ is usually required.

Two maintenance actions

At $t_{i,j}$, the decision maker has the option: replace the system or wait until the next inspection.

- **Corrective replacement**: implement if the system is found to have failed during the inspection, incurring a corrective replacement cost denoted as c_c.
- **Preventive replacement**: implement when it is expected that the system is nearing the failure state, incurring a preventive replacement cost denoted as c_p.

Candidate replacement time at $t_{i,j}$

$$\mathcal{T}_{i,j} = \inf_{T_{i,j}} \left\{ \int_0^{T_{i,j}-t_{i,j}} \frac{c_c + c_i\lfloor x + t_{i,j} \rfloor + c_b}{x + t_{i,j} + \varpi} f_{S_t}(x|y_{i,1:j})dx + \int_{T_{i,j}-t_{i,j}}^{+\infty} f_{S_t}(x|y_{i,1:j}) \frac{c_p + c_i\lfloor T_{i,j} - \varpi \rfloor}{T_{i,j}} dx \right\},$$

where $\lfloor \psi \rfloor = \max\{h \in \mathbb{Z} \mid t_{i,h} \leq \psi\}$, and c_b is the downtime cost during the preparation time after system failure.

Optimal preparation and replacement time

As the values of $\mathcal{T}_{i,j}$ are successively updated,

$$\mathcal{T}_i' = \inf_{t_{i,j}}\{\mathcal{T}_{i,j} - t_{i,j} \leq \varpi\}, \quad \text{and} \quad \mathcal{T}_i^* = \mathcal{T}_i' + \varpi. \tag{26}$$

Performance evaluation

- Consider a set of I systems, each of which operates for a single cycle.
- Let $\mathbb{X}_i = \min\{\mathcal{T}_i^*, \mathcal{T}_i^{\mathsf{f}}\}$, where $\mathcal{T}_i^*$ represents predicted optimal maintenance time, and $\mathcal{T}_i^{\mathsf{f}}$ represents actual failure time.

Actual cost rate of the i-th system

$$CR_i = \begin{cases} \dfrac{c_p + c_i\lfloor \mathbb{X}_i - \varpi \rfloor}{\mathcal{T}_i^*}, & \mathbb{X}_i = \mathcal{T}_i^*, \\ \dfrac{c_c + c_i\lfloor \mathbb{X}_i \rfloor + c_b}{\mathcal{T}_i^{\mathsf{f}} + \varpi}, & \mathbb{X}_i = \mathcal{T}_i^{\mathsf{f}}, \end{cases} \tag{27}$$

Average cost rate for all systems

$$\overline{CR} = \frac{\sum_{i=1}^{I} CR_i}{I}. \tag{28}$$

Algorithm 3: RUL-based adaptive replacement policy.

Input: $y, c_c, c_p, c_b, \varpi, \mathcal{D}, j$.

Output: $\mathcal{T}_i^*$, $CR_i, i = 1, \ldots, I$, and $\overline{CR}$.

for $i = 1$ **to** I **do**
- **while** *no maintenance performed* **do**
  - **if** *the system is operational* **then**
    - Collect new inspection data $Y_{i,j}$;
    - Update model parameter estimates using Bayesian methods in Section Section 3;
    - Compute RUL distribution $\{f_{S_t}(x|y_{i,1:j})\}_{x=0}^{+\infty}$ using (9);
    - Determine $\mathcal{T}_{i,j}$ by (22), and find $\mathcal{T}_i'$ by (23);
    - **if** $t_{i,j} = \mathcal{T}_i'$ **then**
      - Inspection is completed, and preventive maintenance at $\mathcal{T}_i^*$.
    - **end**
  - **end**
  - **else**
    - Corrective maintenance;
    - Set $\mathcal{T}_i^{\mathrm{f}} = t_{i,j}$.
  - **end**
  - $j = j + 1$.
- **end**
- Compute CR_i by (24).

end

Compute $\overline{CR}$ by (25).

Simulation study

Simulation settings

- (I) $I = 5$ and $n_i = 20$; (II) $I = 5$ and $n_i = 40$; (III) $I = 8$ and $n_i = 20$.
- Considering the heterogeneity, we generate $\delta_{1,1}, \ldots, \delta_{1,I}$ from $N(4,1)$, $\delta_{2,1}, \ldots, \delta_{2,I}$ from $N(15,1)$, and $\tau_1, \ldots, \tau_I$ from $N(10,1)$.
- For each scenario, we generate 500 samples to reduce the effects of randomness on the results.

Simulation study

- **Bayesian method**:
  - Flat priors: $(\mu_\tau, \sigma_\tau) \sim NIGa(8, 100, 0.01, 0.01)$, $(\mu_1, \sigma_1) \sim NIGa(1, 100, 0.01, 0.01)$, $(\mu_2, \sigma_2) \sim NIGa(2, 100, 0.01, 0.01)$, and $\gamma \sim N(5, 100)$.
  - Initiate a burn-in period comprising $\mathcal{L} = 5000$ iterations, and an additional $\mathcal{S} - \mathcal{L} = 5000$ iterations are conducted to obtain posterior samples.
- **ML method**: the point estimates are calculated by the EM algorithm, corresponding interval estimates are calculated by parametric bootstrap method with $\mathcal{B} = 500$.
- Indexes of assessing different methods: **relative bias** (RB), **rooted mean squared error** (RMSE) and 95% **coverage probability** (CP).

Parameter estimation performance of two methods

Table 7: Parameter estimation from Bayes and ML methods for two scenarios.

Scen.	Meth.	Stat.	$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{1,4}$	$\delta_{1,5}$	$\delta_{2,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{2,4}$	$\delta_{2,5}$	γ
I	Bayes	RB	0.024	0.029	-0.007	0.015	0.012	-0.026	0.019	0.023	0.056	0.003	0.011
		RMSE	1.326	1.363	1.357	1.332	1.330	0.422	0.424	0.476	0.422	0.431	0.168
		CP	0.956	0.953	0.946	0.953	0.957	0.941	0.925	0.900	0.928	0.926	0.964
	MLE	RB	0.057	0.039	0.040	0.057	0.050	0.065	0.071	0.057	0.078	0.060	0.057
		RMSE	1.315	1.381	1.302	1.401	1.508	0.641	0.645	0.576	0.667	0.739	0.308
		CP	0.889	0.922	0.878	0.900	0.833	0.922	0.922	0.900	0.889	0.867	0.811
Scen.	Meth.	Stat.	$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{1,3}$	$\delta_{1,4}$	$\delta_{1,5}$	$\delta_{2,1}$	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{2,4}$	$\delta_{2,5}$	γ
II	Bayes	RB	-0.005	0.007	0.023	0.011	-0.005	-0.019	0.000	0.016	0.000	0.012	0.001
		RMSE	1.068	1.011	1.065	1.015	1.044	0.349	0.283	0.275	0.355	0.332	0.124
		CP	0.930	0.945	0.950	0.944	0.927	0.902	0.925	0.947	0.885	0.902	0.914
	MLE	RB	0.036	0.035	0.017	0.032	0.039	0.029	0.041	0.036	0.025	0.042	0.039
		RMSE	0.944	1.010	0.880	0.900	0.985	0.331	0.358	0.323	0.328	0.346	0.150
		CP	0.905	0.890	0.905	0.920	0.900	0.895	0.890	0.930	0.930	0.920	0.865

Model comparison in reliability estimation

- Linear $\Lambda(t) = t$; Power $\Lambda(t;\alpha) = t^{\alpha}$; Exponential $\Lambda(t;\alpha) = \exp(\alpha t) - 1$.

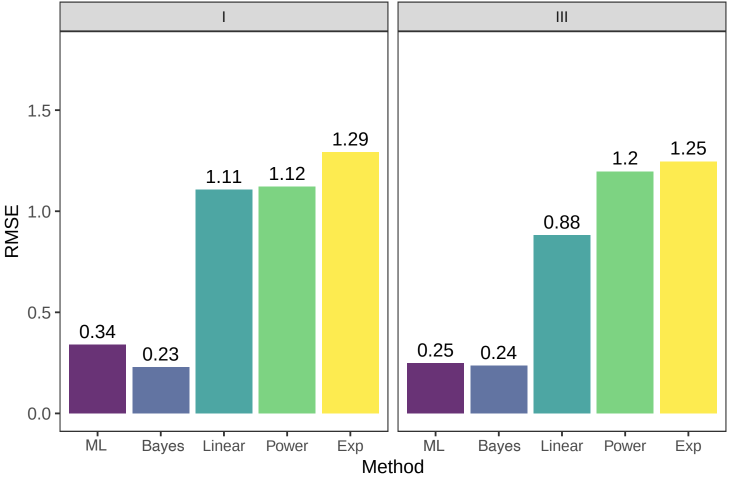


Figure 8: Average RMSE of MTTF estimators based on various models.

Change point estimation under real-time scenarios

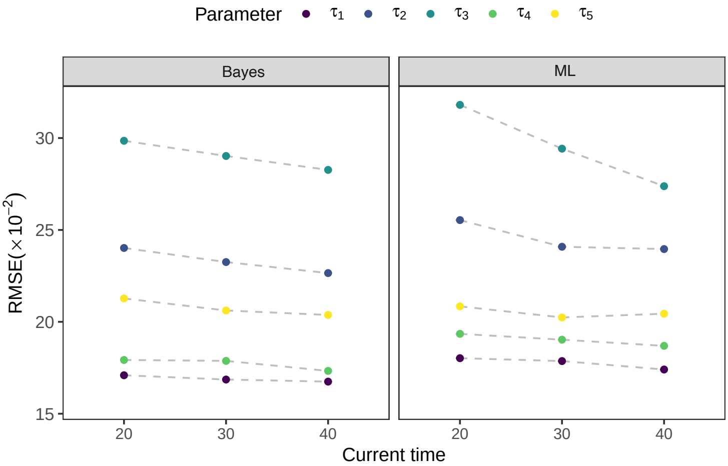


Figure 9: Average RMSE of the change point estimates at three different time points.

Parameter estimation

Table 8: Parameter estimation based on the proposed model.

		HB			ML				HB			ML	
		$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\tau}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$			$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\tau}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$
	2.5%	0.422	2.198	22.257	0.497	2.511		2.5%	0.467	1.993	24.151	0.561	2.120
# 1	Mean	0.532	2.516	23.187	0.510	2.632	# 4	Mean	0.583	2.291	25.008	0.576	2.221
	97.5%	0.645	2.851	24.664	0.518	2.713		97.5%	0.703	2.595	26.060	0.587	2.288
	2.5%	0.523	2.013	24.365	0.638	2.113		2.5%	0.495	2.162	23.184	0.624	2.382
# 2	Mean	0.653	2.312	25.336	0.658	2.215	# 5	Mean	0.621	2.472	24.003	0.642	2.496
	97.5%	0.785	2.615	26.557	0.670	2.282		97.5%	0.752	2.809	25.370	0.654	2.572
	2.5%	0.336	2.161	26.316	0.405	2.412		2.5%	0.464	2.130	24.722	0.559	2.324
# 3	Mean	0.428	2.487	26.761	0.414	2.531	# 6	Mean	0.577	2.443	25.583	0.574	2.440
	97.5%	0.518	2.831	27.381	0.420	2.610		97.5%	0.697	2.769	26.306	0.585	2.517

Table 9: RMSE and RB results for different models.

Model	Training(30)		Prediciton (19)		Overall	
	RMSE	RB	RMSE	RB	RMSE	RB
Proposed	0.448	0.248	1.538	0.060	1.020	0.175
Linear	3.476	1.442	3.685	0.156	3.558	0.943
Power	2.057	0.568	2.475	0.113	2.229	0.391
Exp	0.908	0.313	1.611	0.065	1.230	0.217
Duan	0.434	0.239	1.976	0.075	1.276	0.175

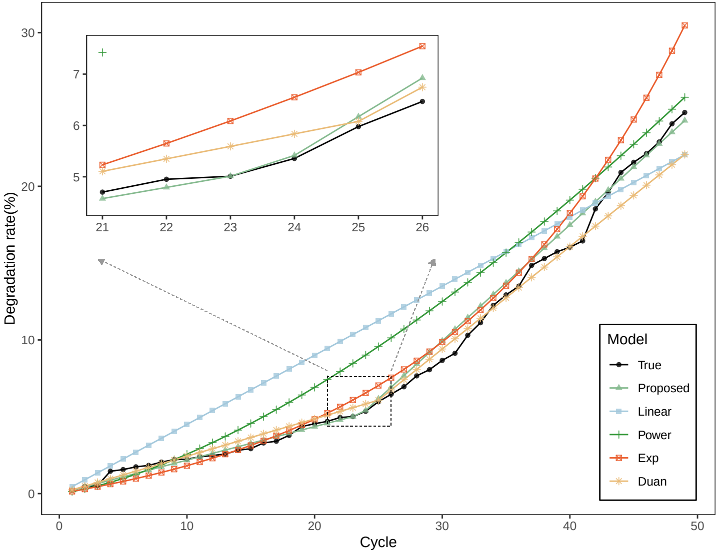


Figure 10: Degradation path training and prediction results for battery #2 using different methods, with a zoomed-in view of the potential change point locations.

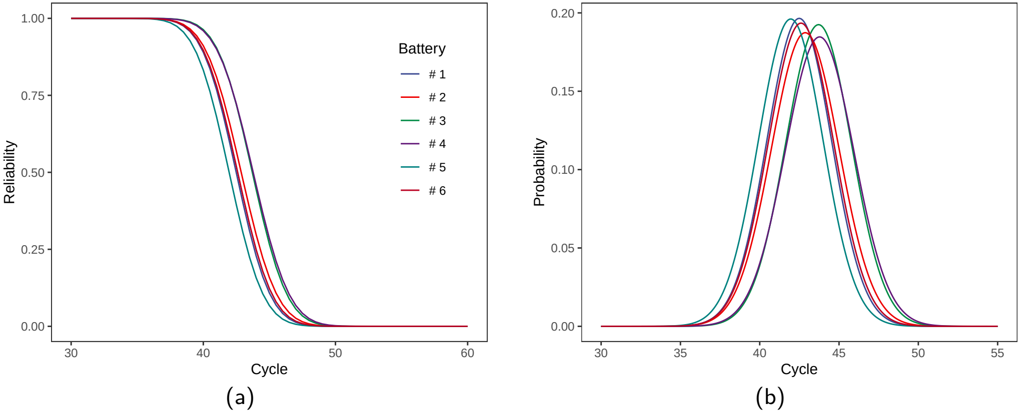
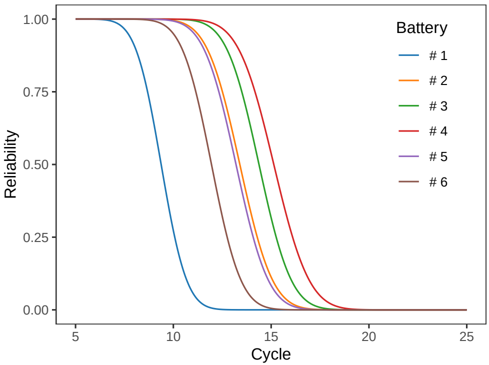
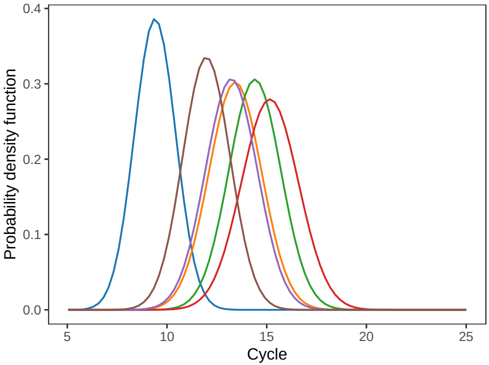


Figure 11: Reliability and density functions of failure time based on HB method.



(a)



(b)

Figure 12: Reliability and density functions of RUL based on HB method.

RUL-based adaptive maintenance policy

1. Cycles 1-30 as historical data, continuously acquiring new data over time.
2. $c_i = 2, c_c = 600, c_p = 200$, and $c_b = 100$.
3. Maintenance preparation period is $\varpi = 1$.

Benchmark policies

i) **Classical replacement policy (CRP)**: preventive maintenance time is determined by the system's mean time to failure $\bar{\mathcal{T}}^F$.

ii) **Ideal replacement policy (IRP)**: the assumption of perfect predicted failure time $\mathcal{T}_i^P$.

Table 10: Candidate preparation time at consecutive data-acquire epochs.

Cycle($\times 300$)	Battery #2			Battery #3		
	Real RUL	MRL	$\mathcal{T}_{2,j}$	Real RUL	MRL	$\mathcal{T}_{3,j}$
31	12	13.865	43	13	13.228	46
33	10	11.219	41	11	10.278	43
35	8	7.624	41	9	8.389	42
37	6	5.986	41	7	6.884	42
39	4	4.040	42	5	4.206	43
41	2	2.764	43	3	2.318	44
42	1	1.235	**43**	2	1.556	44
43	-	-	-	1	0.380	**44**

Optimal preparation times $(42, 43)$; optimal replacement times $(43, 44)$.

Table 11: Maintenance cost rates for 6 batteries under the adaptive replacement policy.

Battery	FC	TS			Linear			Power			Exp		
		$\mathcal{T}_i^*$	Action	CR	$\mathcal{T}_i^*$	Action	CR	$\mathcal{T}_i^*$	Action	CR	$\mathcal{T}_i^*$	Action	CR
1	40	37	P	7.351	37	P	7.351	40	P	6.950	35	P	7.657
2	43	43	P	6.605	42	P	6.714	-	C	17.909	40	P	6.950
3	44	44	P	6.500	44	P	6.500	-	C	17.556	42	P	6.714
4	45	44	P	6.500	43	P	6.605	-	C	17.217	41	P	6.829
5	41	40	P	6.950	39	P	7.077	-	C	18.667	38	P	7.211
6	42	42	P	6.714	41	P	6.829	-	C	18.326	40	P	6.950

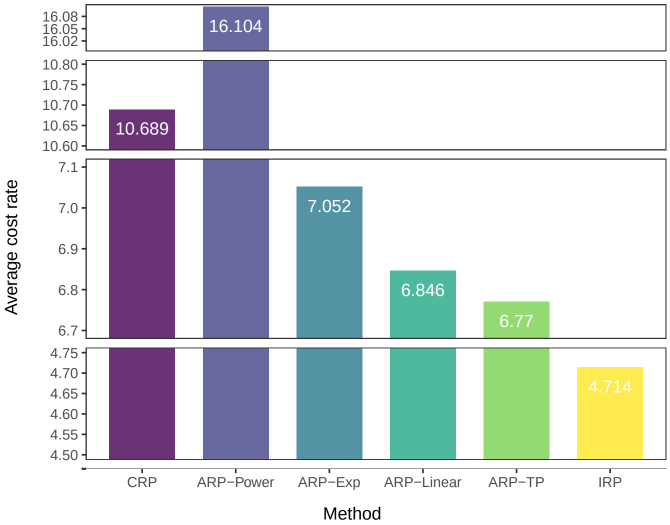


Figure 13: Average cost rate for each policy.

Outline

1 Introduction

2 Two-phase degradation model

3 Multivariate degradation model
- Bivariate Wiener model
- Multivariate inverse Gaussian model

4 Conclusion

Outline

1 Introduction

2 Two-phase degradation model

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4 Conclusion

Motivated example: HMT degradation data

- To maintain the high availability and high efficiency of heavy machine tools, preventive maintenance and system health management are implemented.
- The heavy machine tools (HMT) have two important PCs: the positioning accuracy and the output power.
- HMT fails if the value of the positioning accuracy exceeds the threshold level $\omega_1 = 35$ or the value of the output power exceeds the threshold level $\omega_2 = 120$.

HMT with two PCs

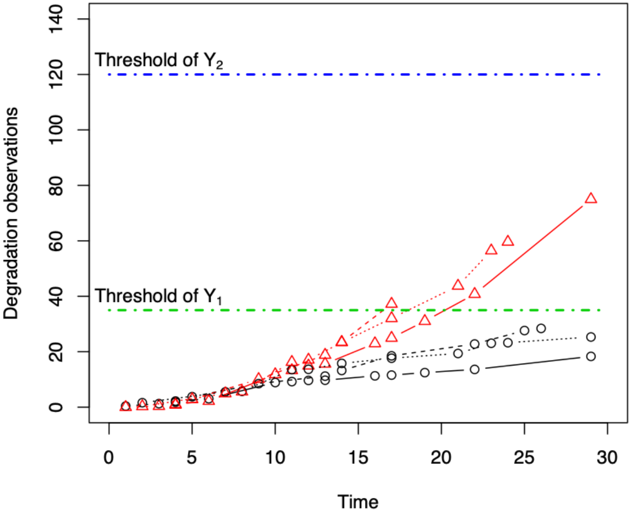


Figure 14: Degradation paths of the positioning accuracy and output power.

Objective

- The positioning accuracy is measured by programmed procedures, while measurements of the output power are recorded by the system operators, and may be missing at some time points.
- Historical information and experts' experience have indicated that these two performance indicators are correlated.

Objective

- How to build a model for bivariate degradation process?
- How to estimate the missing values of the output power?

Related Literature

- LED system consists of many LED lamps for different lighting purposes, and each LED lamp can be viewed as a PC in the LED system (Sari et al., 2009).
- A rubidium discharge lamp: The rubidium consumption and the light intensity (Sun and Balakrishnan, 2013).
- Modeling methods: using copula function (Sun et al. 2010,2012, Wang et al., 2014,2015, Peng et al., 2016, Duan and Wang, 2018).
    - Difficult to choose copula function.
    - Reliability function of product is not analytic.
    - No physical explanation.

Model

Bivariate Wiener degradation model

Assume two PCs in a system, degradation process of the s-th PC is:

$$Y_s(t) = \alpha\beta_s h_s(t,\gamma_s) + \sigma_s B_s(h_s(t,\gamma_s)),\ \ s = 1, 2, \tag{29}$$

- β_s and σ_s denote the drift parameter and the diffusion parameter.
- $h_s(t,\gamma_s)$ is a non-decreasing function of time with $h_s(0,\gamma_s) = 0$.
- $B_s(\cdot)$ is a standard Brownian motion, where $B_1(\cdot)$ and $B_2(\cdot)$ are independent.
- α is random, and follows normal distribution with mean 1 and variance δ^2.

Comments on α

- α could describe the unit-to-unit variation among the systems.
- With the same working environment for both PCs, α is a common factor affecting the degradation process.

Joint PDF of $Y_1(t)$ and $Y_2(t)$

$$\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} \sim \boldsymbol{N}_2(\mu_H, \Sigma), \tag{30}$$

where $\mu_H = \binom{\beta_1 h_1(t,\gamma_1)}{\beta_2 h_2(t,\gamma_2)}$,

$$\Sigma = \begin{pmatrix} \sigma_1^2 h_1(t,\gamma_1) + \delta^2 \beta_1^2 h_1^2(t,\gamma_1) & \delta^2 \beta_1 \beta_2 h_1(t,\gamma_1) h_2(t,\gamma_2) \\ \delta^2 \beta_1 \beta_2 h_1(t,\gamma_1) h_2(t,\gamma_2) & \sigma_2^2 h_2(t,\gamma_2) + \delta^2 \beta_2^2 h_2^2(t,\gamma_2) \end{pmatrix}.$$

Failure-time

- Denote that the threshold level of $Y_s(t)$ is ω_s, $s = 1, 2$.
- The lifetime of the s-th PC is defined as $T_s = \inf\{t : Y_s \geq \omega_s\}$.
- The joint CDF of T_1 and T_2 is

$$F(t_1, t_2) = A_1 + A_2 + A_3 + A_4,$$

where $A_1 = bvn\left(\frac{-\omega_1+\beta_1 h_1(t_1,\gamma_1)}{K_1}, \frac{-\omega_2+\beta_2 h_2(t_2,\gamma_2)}{K_2}, \frac{C_5}{K_1K_2}\right)$,

$$A_2 = \exp\left\{\frac{2\beta_2\omega_2}{\sigma_2^2} + \frac{2\beta_2^2\omega_2^2\delta^2}{\sigma_2^4}\right\} bvn\left(\frac{-\omega_1+\beta_1 h_1(t_1,\gamma_1)+C_1}{K_1}, \frac{-\omega_2-\beta_2 h_2(t_2,\gamma_2)-C_4}{K_2}, \frac{-C_5}{K_1K_2}\right),$$

$$A_3 = \exp\left\{\frac{2\beta_1\omega_1}{\sigma_1^2} + \frac{2\beta_1^2\omega_1^2\delta^2}{\sigma_1^4}\right\} bvn\left(\frac{-\omega_1-\beta_1 h_1(t_1,\gamma_1)-C_3}{K_1}, \frac{-\omega_2+\beta_2 h_2(t_2,\gamma_2)+C_2}{K_2}, \frac{-C_5}{K_1K_2}\right),$$

$$A_4 = \exp\left\{\frac{2\beta_1\omega_1}{\sigma_1^2} + \frac{2\beta_2\omega_2}{\sigma_2^2} + 2\delta^2\left(\frac{\beta_1\omega_1}{\sigma_1^2} + \frac{\beta_2\omega_2}{\sigma_2^2}\right)^2\right\}$$
$$\times bvn\left(\frac{-\omega_1-\beta_1 h_1(t_1,\gamma_1)-C_1-C_3}{K_1}, \frac{-\omega_2-\beta_2 h_2(t_2,\gamma_2)-C_2-C_4}{K_2}, \frac{C_5}{K_1K_2}\right).$$

Failure-time

$$
\begin{aligned}
bvn(x_1,x_2,\theta) &= \frac{1}{2\pi\sqrt{1-\theta^2}}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\exp\left\{-\frac{x^2-2\theta xy+y^2}{2(1-\theta^2)}\right\}\mathrm{d}x\mathrm{d}y,\\
K_1 &= \sqrt{\sigma_1^2 h_1(t_1,\gamma_1)+\beta_1^2\delta^2 h_1^2(t_1,\gamma_1)},\\
K_2 &= \sqrt{\sigma_2^2 h_2(t_2,\gamma_2)+\beta_2^2\delta^2 h_2^2(t_2,\gamma_2)},\\
C_1 &= 2\beta_1 h_1(t_1,\gamma_1)\beta_2\omega_2\delta^2/\sigma_2^2,\\
C_2 &= 2\beta_1\omega_1\beta_2 h_2(t_2,\gamma_2)\delta^2/\sigma_1^2,\\
C_3 &= 2\beta_1^2 h_1(t_1,\gamma_1)\omega_1\delta^2/\sigma_1^2,\\
C_4 &= 2\beta_2^2 h_2(t_2,\gamma_2)\omega_2\delta^2/\sigma_2^2,\\
C_5 &= \beta_1 h_1(t_1,\gamma_1)\beta_2 h_2(t_2,\gamma_2)\delta^2.
\end{aligned}
$$

Failure-time

- The lifetime of system is defined as $T = \min(T_1, T_2)$.

Reliability of system at time t

$$R(t) = F(t,t) + 1 - F_{T_1}(t) - F_{T_2}(t), \tag{31}$$

where $F_{T_s}(t)$ is the CDF of T_s:

$$F_{T_s}(t) = \Phi\left(\frac{\beta_s h_s(t,\gamma_s) - \omega_s}{\sqrt{\beta_s^2\delta^2(h_s(t,\gamma_s))^2 + \sigma_s^2 h_s(t,\gamma_s)}}\right)$$
$$+ \exp\left\{\frac{2\beta_s\omega_s}{\sigma_s^2} + \frac{2\beta_s^2\delta^2\omega_s^2}{\sigma_s^4}\right\}\Phi\left(-\frac{2\beta_s^2\delta^2\omega_s h_s(t,\gamma_s) + \sigma_s^2(\beta_s h_s(t,\gamma_s) + \omega_s)}{\sigma_s^2\sqrt{\beta_s^2\delta^2(h_s(t,\gamma_s))^2 + \sigma_s^2 h_s(t,\gamma_s)}}\right).$$

RUL

- The RULof the $s-$th PC at time t_k is defined as

$$L_{t_k}^{(s)} = \inf\{l : Y_s(l + t_k) \geq \omega_s | Y_s(t_j) < \omega_s, j = 1, 2, \cdots, k\}, \; s = 1, 2,$$

  where $t_1, \; \ldots, \; t_k$ are the measurement times.

- The RUL of the system is defined as

$$L_{t_k} = \min(L_{t_k}^{(1)}, L_{t_k}^{(2)}).$$

- The reliability function of L_{t_k} at time l can be computed as follows:

$$R_{L_{t_k}}(l) = F_{L_{t_k}}(l, l) + 1 - F_{L_{t_k}^{(1)}}(l) - F_{L_{t_k}^{(2)}}(l), \tag{32}$$

$F_{L_{t_k}^{(s)}}(l)$ is the CDF of $L_k^{(s)}$, and its analytical form is

$$
\begin{aligned}
F_{L_{t_k}^{(s)}}(l) = {} & \Phi\left(\frac{\tilde{\mu}\beta_s h_s(l,\gamma_s) - (\omega_s - Y_s(t_k))}{\sqrt{\beta_s^2\tilde{\delta}^2(h_s(l,\gamma_s))^2 + \sigma_s^2 h_s(l,\gamma_s)}}\right) + \exp\left\{\frac{2\tilde{\mu}\beta_s(\omega_s - Y_s(t_k))}{\sigma_s^2}\right. \\
& \left. + \frac{2\beta_s^2\tilde{\delta}^2(\omega_s - Y_s(t_k))^2}{\sigma_s^4}\right\} \\
& \times \Phi\left(-\frac{2\beta_s^2\tilde{\delta}^2(\omega_s - Y_s(t_k))h_s(l,\gamma_s) + \sigma_s^2(\tilde{\mu}\beta_s h_s(l,\gamma_s) + (\omega_s - Y_s(t_k)))}{\sigma_s^2\sqrt{\beta_s^2\tilde{\delta}^2(h_s(l,\gamma_s))^2 + \sigma_s^2 h_s(l,\gamma_s)}}\right).
\end{aligned}
$$

Data

- Suppose that a total of n systems are tested in an experiment.
- For the i-th system, let y_{isj} be the j-th degradation observation of the s-th PC at the measurement time t_{isj}, $s = 1, 2, j = 1, 2, \ldots, m_{is}$.
- $y_{is0} = 0$. Let $z_{isj} = y_{isj} - y_{is(j-1)}$, and $\Lambda_{isj} = h_s(t_{isj}, \gamma_s) - h_s(t_{is(j-1)}, \gamma_s)$, $s = 1, 2, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m_{is}$.
- Then for the i-th system, the model can be described as

$$z_{isj}|\alpha_i \sim \boldsymbol{N}(\alpha_i \beta_s \Lambda_{isj}, \sigma_s^2 \Lambda_{isj}), \text{and } \alpha_i \sim \boldsymbol{N}(1, \delta^2),$$

where $s = 1, 2,\ j = 1, 2, \ldots, m_{is}$.

Bayesian analysis

Prior

- $\beta_s \sim \boldsymbol{N}(1, 10^3); 1/\sigma_s^2 \sim \boldsymbol{IG}(0.01, 0.01); 1/\delta^2 \sim \boldsymbol{IG}(0.01, 0.01); \gamma_s \sim \boldsymbol{IG}(0.01, 0.01).$

Gibbs sampling

- Full conditional posterior distribution of α_i is **normal distribution** with mean $\tilde{\mu}_i$ and variance $\tilde{\delta}_i^2$, where $\tilde{\delta}_i^2 = (\delta^{-2} + \sigma_1^{-2}\beta_1^2 h_1(t_{i1m_{i1}}, \gamma_1) + \sigma_2^{-2}\beta_2^2 h_2(t_{i2m_{i2}}, \gamma_2))^{-1}$, $\tilde{\mu} = \tilde{\delta}_i^2(\delta^{-2} + \sigma_1^{-2}\beta_1 y_{i1m_{i1}} + \sigma_2^{-2}\beta_2 y_{i2m_{i2}})$.
- Full conditional posterior distribution of β_s is **normal distribution** with mean $\tilde{\mu}_{\beta_s}$ and variance $\tilde{\sigma}_{\beta_s}^2$, where

$$\tilde{\sigma}_{\beta_s}^2 = (1/\sigma_{\beta_s}^2 + \sum_{i=1}^{n} \alpha_i^2 h_s(t_{ism_{is}}, \gamma_s)/\sigma_s^2)^{-1},$$

$$\tilde{\mu}_{\beta_s} = \tilde{\sigma}_{\beta_s}^2(\mu_{\beta_s}/\sigma_{\beta_s}^2 + \sum_{i=1}^{n} \alpha_i y_{ism_{is}}/\sigma_s^2), s = 1, 2.$$

Bayesian analysis

Gibbs sampling

- The full conditional posterior distribution of σ_s^2 is **inverse gamma distribution**

$$\boldsymbol{IG}\left(a_s + \sum_{i=1}^{n} m_{is},\ b_s + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m_{is}} {(z_{isj}-\alpha_i\beta_s\Lambda_{isj})^2}/{2\Lambda_{isj}}\right),\quad s = 1, 2.$$

- The full conditional posterior density function of γ_s is proportional to

$$\prod_{i=1}^{n}\prod_{s=1}^{2}\prod_{j=1}^{m_{is}} \frac{1}{\sqrt{\Lambda_{isj}}} \exp\left\{-\frac{(z_{isj}-\alpha_{is}\Lambda_{isj})^2}{2\sigma_s^2\Lambda_{isj}}\right\} (\gamma_s)^{c_s-1} \exp\{-d_s\gamma_s\}.$$

Estimation of the missing values

- If we just observe the degradation value of $Y_1(t_k)$ at the time t_k, estimating the missing value $Y_2(t_k)$ is of our interest.
- Let $\Delta Y_s(t_k) = Y_s(t_k) - Y_s(t_{k-1})$, and $\Delta h_{sk} = h_s(t_k, \gamma_s) - h_s(t_{k-1}, \gamma_s),\ s = 1, 2.$
- We can obtain that

$$\begin{pmatrix} \Delta Y_1(t_k) \\ \Delta Y_2(t_k) \end{pmatrix} \sim \boldsymbol{N}_2(\Delta\mu_H, \Delta\Sigma), \tag{33}$$

where $\Delta\mu_H = \binom{\beta_1 \Delta h_{1k}}{\beta_2 \Delta h_{2k}}$,

$$\Delta\Sigma = \begin{pmatrix} \sigma_1^2 \Delta h_{1k} + \delta^2 \beta_1^2 (\Delta h_{1k})^2 & \delta^2 \beta_1 \beta_2 \Delta h_{1k} \Delta h_{2k} \\ \delta^2 \beta_1 \beta_2 \Delta h_{1k} \Delta h_{2k} & \sigma_2^2 \Delta h_{2k} + \delta^2 \beta_2^2 (\Delta h_{2k})^2 \end{pmatrix}.$$

- Given $\Delta Y_1(t_k)$, the conditional mean of $\Delta Y_2(t_k)$ is

$$\mathsf{E}(\Delta Y_2(t_k)) = \beta_2 \Delta h_{2k} + \frac{\delta^2 \beta_1 \beta_2 \Delta h_{2k}}{\sigma_1^2 + \delta^2 \beta_1^2 \Delta h_{1k}} (\Delta Y_1(t_k) - \beta_1 \Delta h_{1k}).$$

- The Bayesian estimation of $Y_2(t_k)$ can be obtained as

$$\tilde{Y}_2(t_k) = Y_2(t_{k-1}) + \int \left[\beta_2 \Delta h_{2k} + \frac{\delta^2 \beta_1 \beta_2 \Delta h_{2k}}{\sigma_1^2 + \delta^2 \beta_1^2 \Delta h_{1k}} (\Delta Y_1(t_k) - \beta_1 \Delta h_{1k}) \right] f(\Theta|z) \mathrm{d}\Theta,$$

where $f(\Theta|z)$ is the posterior PDF of Θ.

Simulation study

- The mean degradation paths of the two PCs are $1.5t$ and $0.7t^2$. Thus, $(\beta_1, \beta_2) = (1.5, 0.7)$, and $(\gamma_1, \gamma_2) = (1, 2)$.
- The diffusion parameters $(\sigma_1^2, \sigma_2^2) = (0.4, 0.3)$, and $\delta^2 = 0.04$.
- A total number of n systems are put into test, and each systems are measured m times. We choose $n = 3, 4, 5$ and $m = 6, 10$.
- 10,000 independent datasets for each experimental setting are generated to compute the point estimates, the root mean square errors (RMSE) and the empirical coverage probabilities with nominal level 95%.
- We run the Gibbs sampling 80,000 times, and discard the first 20,000 times as the burn-in period. The length of the thinning interval is taken as 20.

Table 12: Bayesian estimates of the parameters based on 10,000 replications.

(n, m)	Estimates	β_1	β_2	σ_1^2	σ_2^2	δ^2	γ_1	γ_2
(3,6)	Mean	1.521	0.718	0.415	0.332	0.0376	1.113	2.215
	RMSE	0.212	0.116	0.175	0.117	0.0239	0.221	0.454
(3,10)	Mean	1.524	0.714	0.413	0.321	0.0382	1.112	2.224
	RMSE	0.205	0.110	0.138	0.101	0.0204	0.201	0.398
(4,6)	Mean	1.518	0.717	0.421	0.322	0.0377	1.095	2.150
	RMSE	0.194	0.105	0.168	0.107	0.0199	0.189	0.361
(4,10)	Mean	1.519	0.711	0.417	0.318	0.0382	1.091	2.121
	RMSE	0.185	0.099	0.128	0.091	0.0178	0.157	0.326
(5,6)	Mean	1.505	0.703	0.416	0.312	0.0386	1.043	2.103
	RMSE	0.167	0.092	0.152	0.089	0.0157	0.146	0.252
(5,10)	Mean	1.506	0.708	0.419	0.308	0.0386	1.051	2.107
	RMSE	0.150	0.085	0.113	0.085	0.0141	0.102	0.228

Table 13: Coverage probabilities of the interval estimates with nominal level 95%.

(n, m)	β_1	β_2	σ_1^2	σ_2^2	δ^2	γ_1	γ_2
(3,6)	0.912	0.913	0.974	0.979	0.934	0.969	0.982
(3,10)	0.926	0.924	0.976	0.977	0.938	0.965	0.975
(4,6)	0.928	0.922	0.974	0.980	0.940	0.968	0.977
(4,10)	0.931	0.938	0.969	0.976	0.936	0.963	0.968
(5,6)	0.934	0.934	0.969	0.976	0.948	0.964	0.962
(5,10)	0.944	0.942	0.968	0.961	0.948	0.959	0.963

Misspecification

- There might be a mis-specification of the distribution α. Another simulation is used to check the robustness of the normal assumption.
- We assume that α follows the normal, lognormal, Weibull and Gamma distributions.
- The proposed model is used to fit data generated under these distributions.
- The estimated 10% quantile of the failure time distribution is compared with the true quantile.
- The relative biases (RB) are computed using 10,000 Monte Carlo replications.

Table 14: RBs of the estimated 10% quantile under different distributions of α.

(n, m)	Normal	Lognormal	Weibull	Gamma
(3,6)	0.389	0.338	0.417	0.407
(3,10)	0.218	0.0248	0.144	0.096
(4,6)	0.0811	0.168	0.306	0.217
(4,10)	0.124	0.0421	0.0584	0.030
(5,6)	0.185	0.0960	0.178	0.0578
(5,10)	0.0400	0.0191	0.0758	0.0377

Case study

- Following Peng et al. (2016), we assume $h_1(t, \gamma_1) = t$ and $h_2(t, \gamma_2) = t^{\gamma_2}$.

Table 15: Bayesian estimation of model parameters using heavy machine tool data.

	Our model			Peng et al. (2016)		
Parameters	Mean	SD	95% CI	Mean	SD	95% CI
β_1	0.871	0.026	(0.826, 0.927)	0.875	0.132	(0.675, 1.172)
β_2	0.142	0.040	(0.079, 0.233)	0.162	0.051	(0.086, 0.281)
σ_1^2	0.951	0.164	(0.683, 1.322)	×	×	×
σ_2^2	0.101	0.037	(0.050, 0.193)	×	×	×
γ_2	1.915	0.091	(1.741, 2.092)	1.867	0.091	(1.690, 2.045)
δ^2	0.0084	0.0096	(0.0019, 0.030)	×	×	×

Table 16: Prediction of the missing degradation observations.

Parameters	Our model			Peng et al. (2016)		
	Mean	SD	95% CI	Mean	SD	95% CI
$y_2(t_{1,11})$	76.09	3.38	(69.95, 83.32)	72.30	3.56	(65.64, 79.75)
$y_2(t_{2,9})$	57.05	1.58	(54.12, 60.33)	56.26	4.64	(48.71, 67.02)
$y_2(t_{2,10})$	71.52	1.84	(68.10, 75.38)	69.66	6.33	(59.03, 83.98)
$y_2(t_{2,11})$	76.73	2.08	(72.55, 80.76)	74.45	6.90	(62.82, 90.03)
$y_2(t_{3,11})$	85.52	2.49	(80.94, 90.79)	84.43	5.53	(75.19, 96.88)

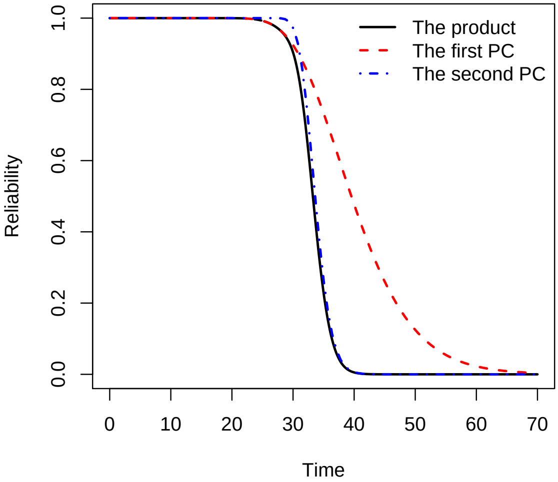


Figure 15: The reliability of the system and the two PCs.

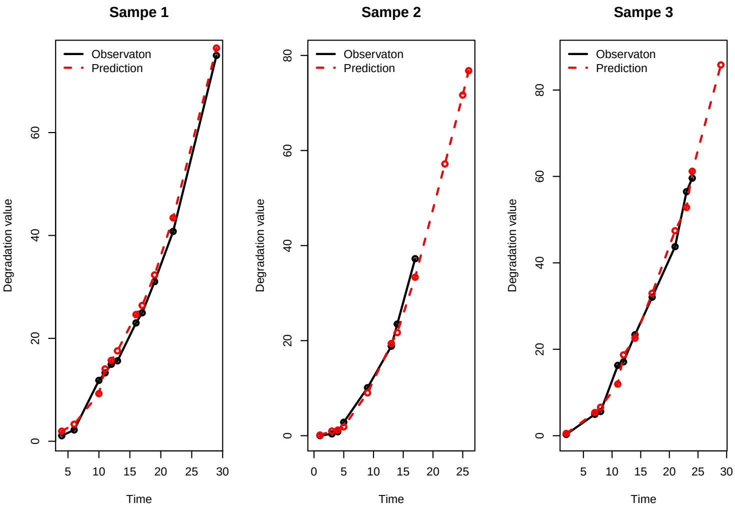


Figure 16: Estimation of degradation values of the second PC.

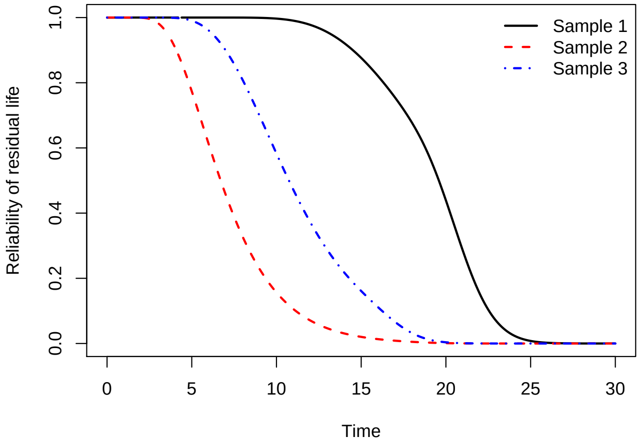


Figure 17: The reliability functions of the RUL for the three systems.

Outline

1 Introduction

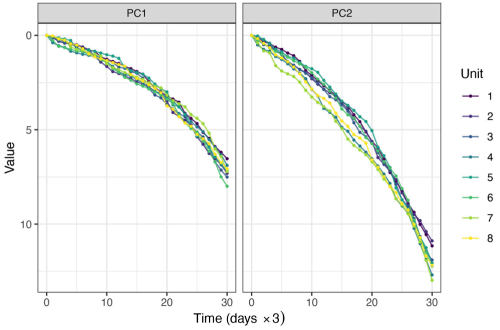
2 Two-phase degradation model

3 Multivariate degradation model

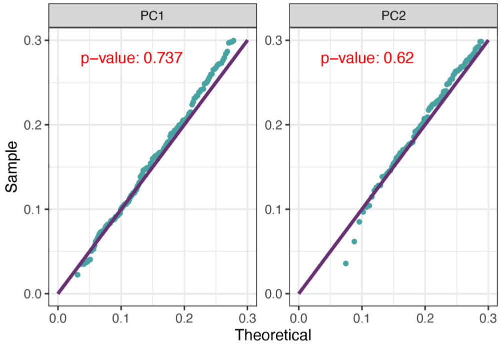
- Bivariate Wiener model
- Multivariate inverse Gaussian model

4 Conclusion

Motivated example: PMB degradation data



(a) Degradation paths



(b) Q-Q plots using IG distribution

Figure 18: Summary of Permanent magnet brake (PMB) data for two PCs: degradation paths and Q-Q plots.

PMB data with two PCs

Unit	1	2	3	4	5	6	7	8
Correlation	0.819	0.749	0.806	0.840	0.779	0.749	0.765	0.800

Figure 19: Pearson correlation coefficients of two PCs across various units.

- **Objective**: establish a multivariate IG process model incorporating common effects.

Related literature

Multivariate degradation modeling

- Copula-based method
- Multivariate distribution-based method
- Common-effect-based method
  a) Frailty model-based method
  b) Stochastic process summation method

Challenges

1. **Copula-based method**: Faces difficulties in selecting appropriate copulas and providing clear physical interpretations.
2. **Multivariate distribution-based method**: Mostly limited to bivariate cases, with challenges in extending to multivariate distributions.
3. **Frailty model-based method**: The use of a single frailty factor limits the model's generality.

Advantages of stochastic process summation

- Model parameters increase linearly with dimensionality, simplifying high-dimensional degradation modeling.

Contributions

(i) Construct a multivariate rIG process using the common-effect method and analyze its properties and system lifetime distribution.

(ii) Apply Gauss-Legendre (GL) quadrature for approximating the complex integral in the lifetime distribution.

(iii) Use the EM algorithm for parameter estimation, with parametric bootstrap for confidence intervals.

Model definition

Degradation process of the k-th PC

$$Y_k(t) = X_k(t) + Z(t), k = 1, \ldots, K, \tag{34}$$

where $Z(t) \sim r\mathcal{IG}\left(\Lambda_0(t), \gamma\right)$ and $X_k(t) \sim r\mathcal{IG}\left(\Lambda_k(t), \gamma\right)$ are independent of each other, $\Lambda_k(t)$ and $\Lambda_0(t)$ are monotonically increasing functions of t.

Based on the additive property of the rIG distribution, $Y_k(t)$ is

$$Y_k(t) \sim rIG\left(\Lambda_k(t) + \Lambda_0(t), \gamma\right), k = 1, \ldots, K. \tag{35}$$

Proposition 1

The mean and variance of the degradation process $Y_k(t)$ are

$$\mathrm{E}\left[Y_k(t)\right] = \frac{\Lambda_0(t) + \Lambda_k(t)}{\gamma}, \quad \text{and} \quad \mathrm{Var}\left[Y_k(t)\right] = \frac{\Lambda_0(t) + \Lambda_k(t)}{\gamma^3}, \tag{36}$$

respectively. Meanwhile, the common effect $Z(t)$ introduces dependence among the multiple degradation processes

$$\mathrm{Cov}\left[Y_{k_1}\left(t_1\right), Y_{k_2}\left(t_2\right)\right] = \frac{\min\left(\Lambda_0\left(t_1\right), \Lambda_0\left(t_2\right)\right)}{\gamma^3}, k_1 \neq k_2. \tag{37}$$

At any given time t, Pearson correlation coefficient is

$$\rho\left[Y_{k_1}(t), Y_{k_2}(t)\right] = \frac{\Lambda_0(t)}{\sqrt{\left(\Lambda_0(t) + \Lambda_{k_1}(t)\right)\left(\Lambda_0(t) + \Lambda_{k_2}(t)\right)}}, k_1 \neq k_2. \tag{38}$$

Proposition 2: Joint PDF and CDF of $Y_1(t), \ldots, Y_K(t)$

$$f_{Y(t)}\left(y_1, \ldots, y_K\right) = \int_0^{\tilde{y}} f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \prod_{k=1}^{K} f_{rIG}\left(y_k - z; \Lambda_k(t), \gamma\right) \mathrm{d}z,$$

where $\tilde{y} = \min\{y_1, \ldots, y_K\}$, where $y_1, \ldots, y_K$ are the observed dagradation values, $f_{rIG}(\cdot)$ is given by (9). The CDF is expressed as

$$F_{Y(t)}\left(y_1, \ldots, y_K\right) = \int_0^{\tilde{y}} f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \prod_{k=1}^{K} F_{rIG}\left(y_k - z; \Lambda_k(t), \gamma\right) \mathrm{d}z,$$

where $F_{rIG}(\cdot)$ is given by (10).

System failure-time

Let $T_{\boldsymbol{\mathcal{D}}} = \inf\{t : Y_1(t) \geq \mathcal{D}_1$ or $\cdots$ or $Y_K(t) \geq \mathcal{D}_K\}$, where $\boldsymbol{\mathcal{D}} = (\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_K)'$ is a vector storing all PC failure thresholds.

Proposition 3: CDF of system failure time $T_{\boldsymbol{\mathcal{D}}}$

$$F_{T_{\boldsymbol{\mathcal{D}}}}(t \mid \boldsymbol{\Lambda}(t), \gamma, \boldsymbol{\mathcal{D}}) = \int_0^{\tilde{y}} \left[1 - \prod_{k=1}^{K} \left(F_{rIG}(\mathcal{D}_k - z; \Lambda_k(t), \gamma)\right)\right] f_{rIG}\left(z; \Lambda_0(t), \gamma\right) \mathrm{d}z,$$

where $\boldsymbol{\Lambda}(t) = (\Lambda_0(t), \ldots, \Lambda_K(t))'$, and $\tilde{y} = \min\{y_1, \ldots, y_K\}$.

Integral approximation

GL quadrature method

CDF of system failure-time can be approximated as

$$F_{T_{\boldsymbol{\mathcal{D}}}}(t \mid \boldsymbol{\Lambda}(t), \gamma, \boldsymbol{\mathcal{D}})$$
$$\approx \frac{\tilde{y}}{2} \sum_{q=1}^{l} w_q \left[1 - \prod_{k=1}^{K} \left(F_{rIG}(\mathcal{D}_k - \frac{\tilde{y}(u_q+1)}{2}; \Lambda_k(t), \gamma)\right)\right] f_{rIG}\left(\frac{\tilde{y}(u_q+1)}{2}; \Lambda_0(t), \gamma\right).$$

where l is a given order, u_q is the root of the Legendre polynomial and w_q is the corresponding weight.

Data

- n systems are tested in an experiment.
- The degradation of the K PCs in the i-th system are measured at m_i time points, denoted as $\boldsymbol{T}_i = (t_{i,1}, \ldots, t_{i,m_i})'$,
- Degradation values are $\boldsymbol{Y}_{i,k} = (Y_{i,k,1}, \ldots, Y_{i,k,m_i})'$ for $k = 1, \ldots, K$, $i = 1, \ldots, n$.
- The degradation increments of the k-th PC between $(t_{i,j-1}, t_{i,j}]$ as $\Delta Y_{i,k,j} \triangleq Y_{i,k,j} - Y_{i,k,j-1}$ for $j = 1, \ldots, m_i$.
- Set $t_{i,0} = 0$ and $Y_{i,k,0} = 0$.
- Denote $\Delta \boldsymbol{Y}_{i,:,j} = (\Delta Y_{i,1,j}, \ldots, \Delta Y_{i,K,j})'$.

Parameter

- $\Lambda_k(t) = \Lambda_k(t; \alpha_k, \beta_k)$ involves unknown parameters α_k and β_k, where $k = 0, \ldots, K$.
- Power-law form $\Lambda_k(t) = \beta_k t^{\alpha_k}$ and log-linear form $\Lambda_k(t) = \beta_k \left[\exp(\alpha_k t) - 1\right]$.
- For parameter nonidentifiability problem, we assume $\Lambda_0(t) = \Lambda_0(t; \alpha_0)$.
- Let $\mathbf{\Lambda}(t) = (\Lambda_0(t), \ldots, \Lambda_K(t))'$.
- Model parameters are $\boldsymbol{\theta} = \{\boldsymbol{\beta}, \boldsymbol{\alpha}, \gamma\}$, with $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_K)'$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_K)'$.

Likelihood

Given the observed data $\Delta\boldsymbol{Y}_{i,:,j}, i = 1, \ldots, n, j = 1, \ldots, m_i$,

Likelihood function

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \sum_{j=1}^{m_i} \ln p\left(\Delta\boldsymbol{Y}_{i,:,j} \mid \boldsymbol{\theta}\right)$$

with

$$p\left(\Delta\boldsymbol{Y}_{i,:,j} \mid \boldsymbol{\theta}\right) = \int_0^{\Delta\tilde{y}_{i,j}} f_{rIG}\left(\Delta z_{i,j}; \Delta\Lambda_0(t_{i,j}), \gamma\right) \times \prod_{k=1}^{K} f_{rIG}\left(\Delta y_{i,k,j} - \Delta z_{i,j}; \Delta\Lambda_k(t_{i,j}), \gamma\right) \mathrm{d}\Delta z_{i,j}, \tag{39}$$

where $\Delta\tilde{y}_{i,j} = \min\{\Delta Y_{i,1,j}, \ldots, \Delta Y_{i,K,j}\}$.

EM algorithm

- Consider $Z_{i,j} = Z_i\left(t_{i,j}\right)$ for $i = 1, \ldots, n, j = 1, \ldots, m_i$ as the missing data;
- Define $\boldsymbol{Z}_i = (Z_{i,1}, \ldots, Z_{i,m_i})'$, $\Delta\Lambda_{i,k,j} = \Delta\Lambda_k(t_{i,j}), k = 0, \ldots, K$.
- $\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \boldsymbol{Z}_i \sim r\mathcal{IG}\left(\Delta\Lambda_{i,k,j}, \gamma\right)$, with $0 \leq \Delta Z_{i,j} \leq \Delta\tilde{y}_{i,j}$.
- Denote $\mathbb{Y} = \{\Delta\boldsymbol{Y}_{i,:,j}, i = 1, \ldots, n, j = 1, \ldots, m_i\}$, and $\mathbb{Z} = \{\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_n\}$.

EM algorithm

Log-likelihood with the complete data

$$\ell(\boldsymbol{\theta} \mid \mathbb{Y}, \mathbb{Z}) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \left\{ \sum_{k=1}^{K} \ln p\left(\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \Delta Z_{i,j}\right) + \ln p\left(\Delta Z_{i,j}\right) \right\}, \tag{40}$$

$$\begin{aligned} \ln p\left(\Delta Y_{i,k,j} - \Delta Z_{i,j} \mid \Delta Z_{i,j}\right) = & -\frac{1}{2}\ln(2\pi) + \ln \Delta\Lambda_{i,k,j} - \frac{3}{2}\ln\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right) \\ & + \gamma \Delta\Lambda_{i,k,j} - \frac{\Delta\Lambda_{i,k,j}^2}{2\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right)} - \frac{\gamma^2\left(\Delta Y_{i,k,j} - \Delta Z_{i,j}\right)}{2}, \end{aligned}$$

$$\ln p\left(\Delta Z_{i,j}\right) = -\frac{1}{2}\ln(2\pi) + \ln \Delta\Lambda_{i,0,j} + \gamma\Delta\Lambda_{i,0,j} - \frac{3}{2}\ln \Delta Z_{i,j} - \frac{\Delta\Lambda_{i,0,j}^2}{2\Delta Z_{i,j}} - \frac{\gamma^2 \Delta Z_{i,j}}{2}.$$

EM algorithm

- **Initialization**: Start with initial values $\boldsymbol{\theta}^{(\mathbf{0})}$ for the parameters $\boldsymbol{\theta}$, and set the tolerance error ω.
- **E-step**: Calculate $Q\left(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(s)}\right)=\mathrm{E}\left[\ell(\boldsymbol{\theta} \mid \mathbb{Y}, \mathbb{Z}) \mid \mathbb{Y}, \boldsymbol{\theta}^{(s)}\right]$, based on the s-th iteration of parameters estimation $\boldsymbol{\theta}^{(s)}$.
- **M-step**: Compute the $(s+1)$-th parameter estimation $\boldsymbol{\theta}^{(s+1)}$ using $\boldsymbol{\theta}^{(s+1)}=\arg\max_{\boldsymbol{\theta}} Q\left(\boldsymbol{\theta} \mid \boldsymbol{\theta}^{(s)}\right)$.
- **Iteration**: Iterate through the E-step and M-step until $\left\|\boldsymbol{\theta}^{(s+1)}-\boldsymbol{\theta}^{(s)}\right\|<\omega$, where $\|\cdot\|$ denotes the Euclidean distance.
- **Output**: Obtain the ML estimates of $\boldsymbol{\theta}$.

Determine initial parameter estimators

1. Based on $\Delta\bar{Y}_{:,k,j} = 1/n\sum_{i=1}^{n}\Delta Y_{i,k,j}$, $\Delta s^2_{:,k,j} = \sum_{i=1}^{n}\left(\Delta Y_{i,k,j} - \Delta\bar{Y}_{:,k,j}\right)^2/(n-1)$, we calculate the estimate for γ:

$$\hat{\gamma} = \sqrt{\frac{\sum_{k=1}^{K}\sum_{j=1}^{m_i}\Delta\bar{Y}_{:,k,j}}{\sum_{k=1}^{K}\sum_{j=1}^{m_i}\Delta s^2_{:,k,j}}}.$$

2. Assuming $\hat{\gamma}$ is known, we optimize the formula to estimate $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$.

$$\begin{aligned}\Psi &= \arg\min_{\boldsymbol{\beta},\boldsymbol{\alpha}} \sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=1}^{K}\frac{\left(\mathrm{E}\left[\Delta Y_{i,k,j}\right] - \Delta Y_{i,k,j}\right)^2}{\mathrm{var}\left[\Delta Y_{i,k,j}\right]}\\ &= \arg\min_{\boldsymbol{\beta},\boldsymbol{\alpha}} \hat{\gamma}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\sum_{k=1}^{K}\left[\frac{\hat{\gamma}^2\Delta Y^2_{i,k,j}}{\Delta\Lambda_{i,0,j} + \Delta\Lambda_{i,k,j}} + \Delta\Lambda_{i,0,j} + \Delta\Lambda_{i,k,j}\right].\end{aligned}$$

Model validation

- **Goodness of fit (GOF) test**: Evaluates each PC's rIG model using χ_1^2 Q-Q plots and the KS test based on the statistic $[\hat{\gamma}\Delta Y_{i,j,k} - \Delta\hat{\Lambda}_k(t_{i,j}) - \Delta\hat{\Lambda}_0(t_{i,j})]^2/\Delta Y_{i,j,k}$, which approximates an i.i.d. χ_1^2 distribution.
- **Common dispersion parameter γ test**: Analyzes if all PCs operate under a common γ or distinct γ_i for each PC through the chi-square test statistic $\tau = -2(\ell_1 - \ell_2)$, contrasting the log-likelihoods of a unified model against a heterogeneous model.
- **Model selection**: Uses the Akaike Information Criterion (AIC), $\mathrm{AIC} = 2\kappa - 2\ell$, to determine the most suitable model, focusing on the trade-off between model fit and complexity.

Simulation study

- A multivariate rIG process with common effects with three PCs, i.e., $K = 3$.

Table 17: Four combinations of $\Lambda_k(t)$ and $\Lambda_0(t)$ with corresponding parameters.

Scen.	$\Lambda_k(t)$	$\Lambda_0(t)$	$\boldsymbol{\alpha}'$	$\boldsymbol{\beta}'$	γ	$\boldsymbol{\mathcal{D}}$
I	$\beta_k t^{\alpha_k}$	t^{α_0}	$(1, 0.8, 1, 1.2)$	$(0.8, 1, 1.2)$	4	$(3.6, 4.8, 7.2)$
II	$\beta_k t^{\alpha_k}$	$\exp(\alpha_0 t) - 1$	$(1, 0.25, 0.33, 0.37)$	$(0.8, 1, 1.2)$	4	$(4, 8, 13)$
III	$\beta_k\left[\exp(\alpha_k t) - 1\right]$	t^{α_0}	$(0.05, 0.3, 0.4, 0.4)$	$(0.8, 1, 1.2)$	4	$(0.44, 0.66, 0.88)$
IV	$\beta_k\left[\exp(\alpha_k t) - 1\right]$	$\exp(\alpha_0 t) - 1$	$(0.1, 0.1, 0.1, 0.1)$	$(0.8, 1, 1.2)$	4	$(0.42, 0.56, 0.42)$

- n units is measured at the same time intervals, and all m_i are equal.
- Three unit sizes n: $5, 8$, and 10. For each configuration, we perform 500 replications.

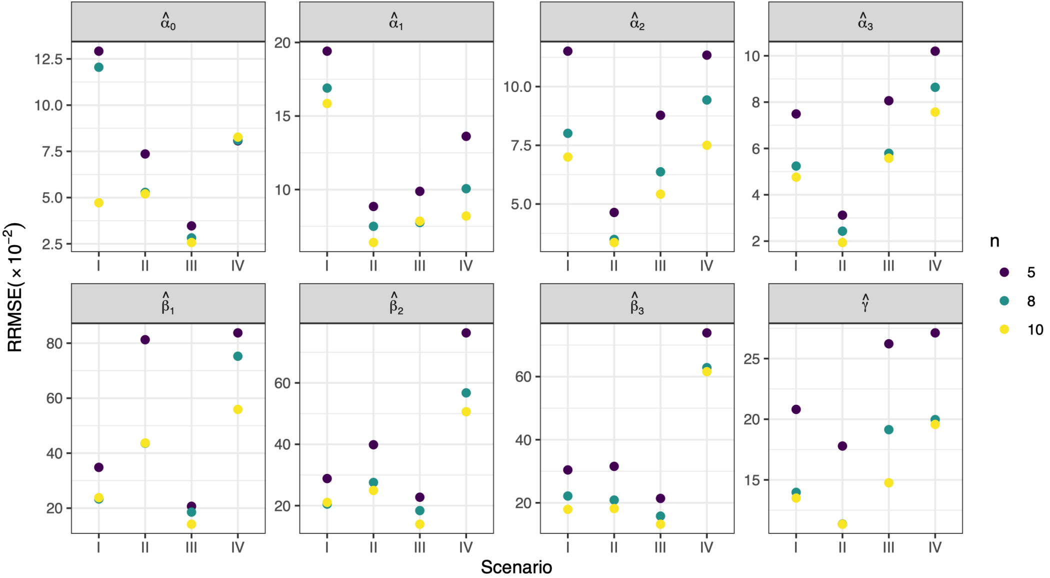


Figure 20: RRMSE $(\times 10^{-2})$ of estimators across different unit sizes and scenarios.

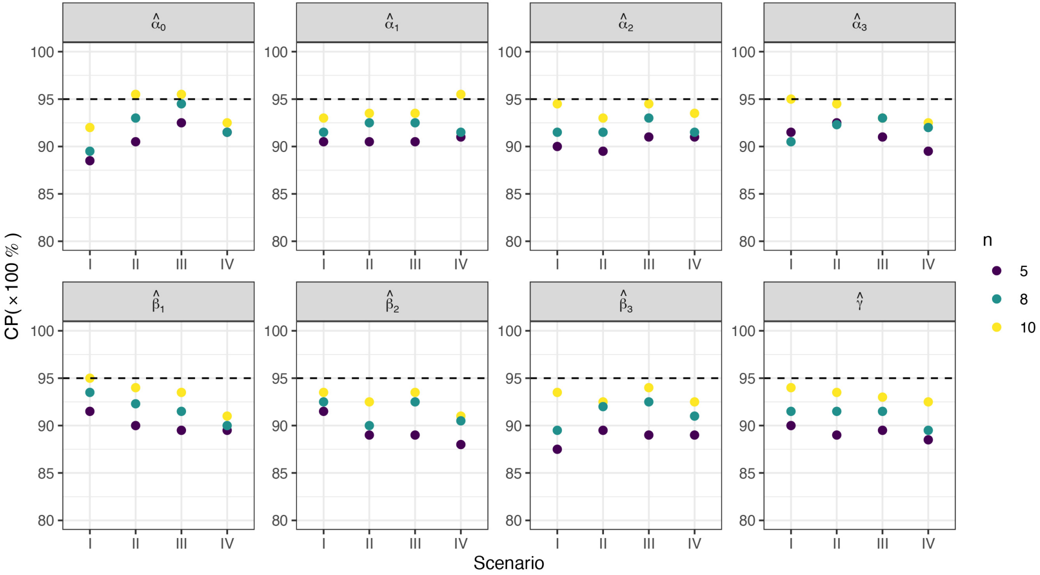


Figure 21: CP $(\times 100\%)$ of estimators across different unit sizes and scenarios.

Performance of reliability estimation

Mean time to failure of the system: $\text{MTTF} = \int_0^\infty 1 - F_{T_{\mathcal{D}}}(t \mid \boldsymbol{\beta}, \boldsymbol{\alpha}, \gamma, \boldsymbol{\mathcal{D}})\mathrm{d}t.$

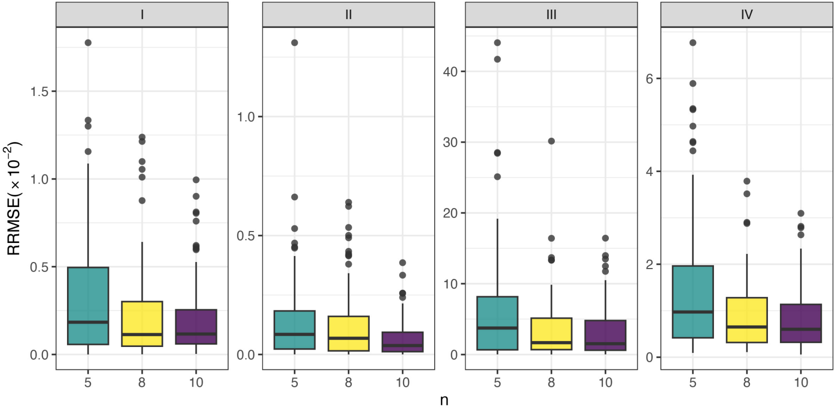


Figure 22: RRMSE $(\times 10^{-2})$ of MTTF estimators across different unit sizes and scenarios.

PMB degradation data

Table 18: Parameter point estimates regarding the PMB data.

Model	Scen.	α_0	α_1	α_2	β_1	β_2	γ	AIC
Proposed	I	0.866	1.296	1.463	0.028	0.124	3.030	-2219.427
	II	0.724	0.104	0.068	0.942	6.182	4.375	-2494.123
	III	0.100	0.994	1.205	0.395	0.531	4.263	**-2603.588**
	IV	0.098	0.009	0.025	42.099	30.476	4.299	-2594.714
Independent	Power	-	1.518	1.456	0.151	0.317	3.703	-637.266
	Log-linear	-	0.056	0.054	6.830	11.944	4.079	-744.887

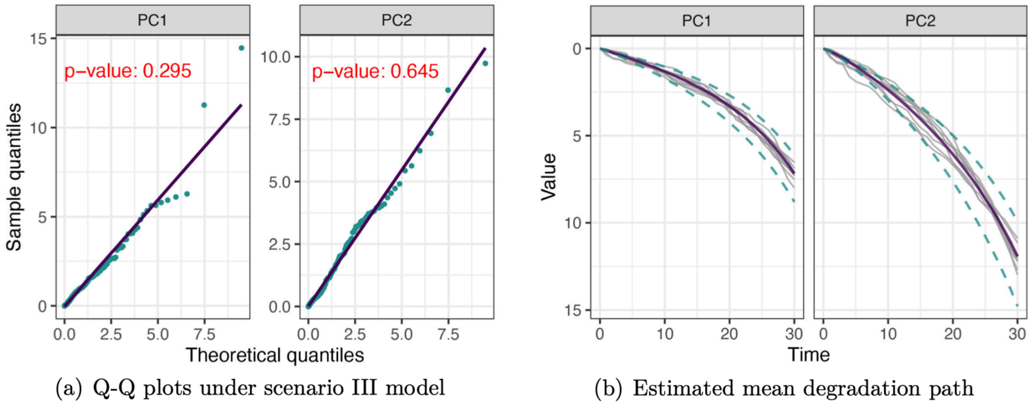
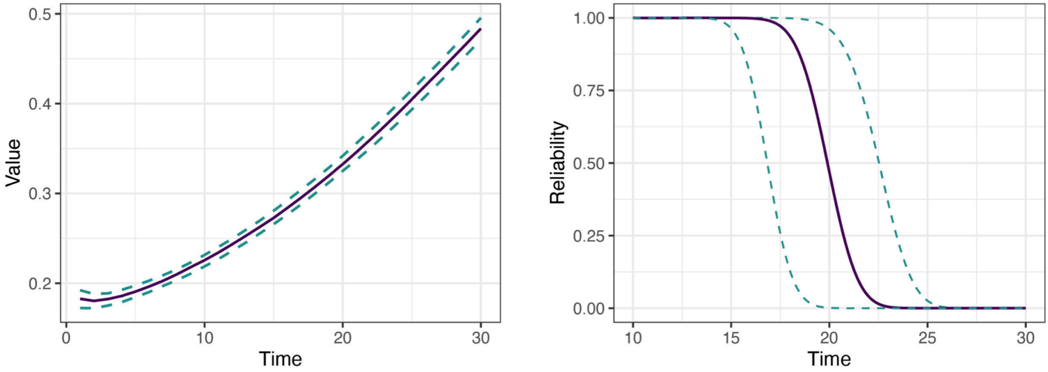


Figure 23: Summary of PMB data analysis results: Q-Q plots under scenario III model and the estimated mean degradation path.



(a) Correlation coefficients　(b) Reliability curves

Figure 24: Correlation coefficients and reliability curves for PMB data.

Fatigue crack-size data

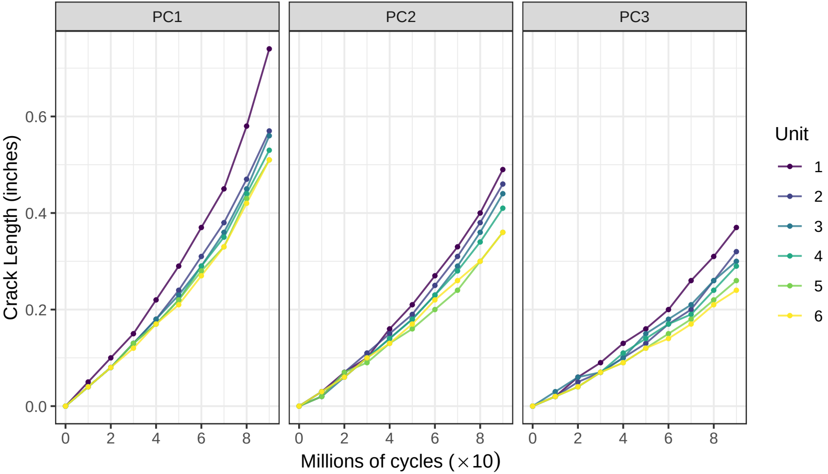
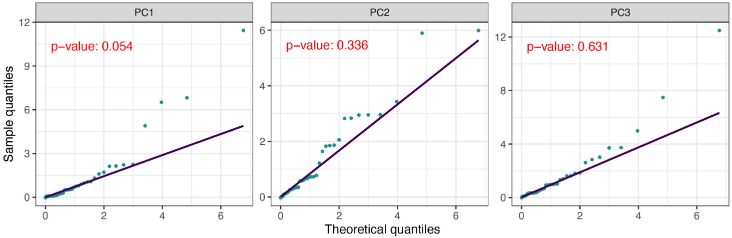


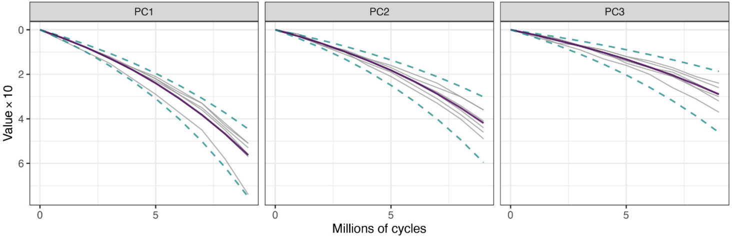
Figure 25: Degradation paths for fatigue crack-size growth data.

Table 19: Parameter point estimates regarding the fatigue crack-size data.

Model	Scen.	α_0	α_1	α_2	α_3	β_1	β_2	β_3	γ	AIC
Dependent	I	1.178	1.327	1.332	0.736	0.796	0.415	0.249	4.836	-717.838
	II	0.957	0.155	0.161	0.162	9.828	6.094	3.429	6.648	-804.636
	III	0.249	1.201	1.153	0.946	1.999	1.490	1.310	6.412	-822.131
	IV	0.067	0.119	0.111	0.090	19.683	16.286	15.236	6.789	**-1410.667**
Independent	Power	-	1.479	1.359	1.206	1.129	1.119	1.107	5.254	-221.197
	Log-linear	-	0.126	0.105	0.081	17.880	17.698	17.978	6.602	-281.749



(a) Q-Q plots under scenario IV model



(b) Estimated mean degradation path

Figure 26: Summary of fatigue crack-size data analysis results: Q-Q plots under scenario IV and the estimated mean degradation path.

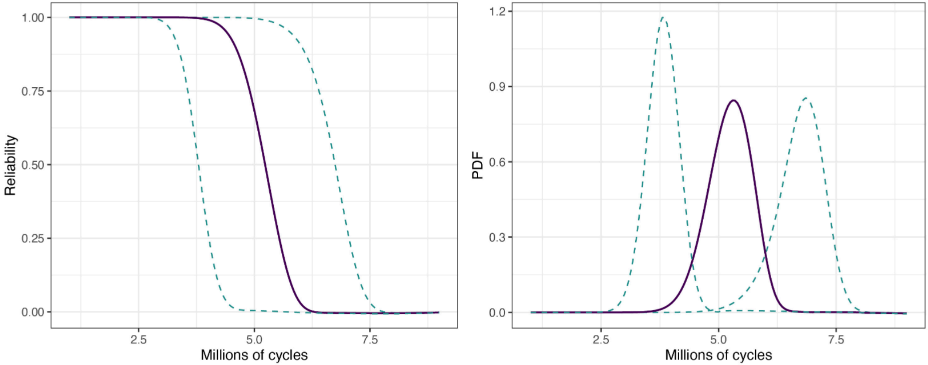


Figure 12: Reliability function and PDF for fatigue crack-size data.

Outline

1 Introduction

2 Two-phase degradation model

3 Multivariate degradation model

4 Conclusion

Conclusion

Degradation modeling

- Two-phase degradation model
  - Wiener model
  - Inverse Gaussian model
- Multivariate degradation model
  - Bivariate Wiener model
  - Multivariate inverse Gaussian model

Thanks!