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Data analysis of progressive-stress accelerated life tests with group effects

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ABSTRACT

Progressive-stress accelerated life testing (PSALT) is a special type of experiment that tests the lifetime of a product with continuously varying stress levels. Due to the limitations of testing equipments and costs, the lifetime data collected by PSALT are usually censored and have group effects. In order to deal with the two characteristics in the data, this paper presents a novel PSALT model with group effects under progressive censoring. Two-stage and Gauss-Hermite quadrature methods are proposed to estimate the model parameters, while the interval estimates are constructed by bootstrap and the asymptotic theorem, respectively. Simulation studies are conducted to compare the proposed model with the traditional models without group effects in terms of the relative bias and root mean squared error under different scenarios. The results show that the proposed model can detect group-to-group variation, and that the models without group effects will result in large biases for estimating the characteristic lifetime of the product. Finally, the proposed model is validated by a real dataset.

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Progressive-stress accelerated life test; two-stage method; gauss–hermite quadrature; progressive censoring

1. Introduction

1.1. Background

Life tests are essential for assessing the reliability of electrical, mechanical, and medical devices, etc. With the development of advanced technologies, traditional lifetime tests are no longer suitable because modern assets are highly reliable. Accelerated life testing (ALT) is a way to solve this problem by increasing certain environmental stresses to collect failures in a shorter period of time. By using ALT, the manufacturers can provide data showing how well a product works, how long it will last and how it will fail in the future. Determining a product's life expectancy before it goes into production will prevent frustration and unnecessary additional warranty costs, which can reduce a company's financial bottom line. The type of ALT can be classified according to its stress loading, for instance, constant stress, step stress, progressive stress, and cyclic stress (Nelson, 2009).

Among them, progressive stress ALT (PSALT) has the highest efficiency to shorten the lifetime of the assets and is flexible to be implemented. The stress loading is continuously increasing, which can expedite the test unit to fail and further reduces the total testing time. To the best of our knowledge, (Prot, 1948) was the first to study this type of stress loading on fatigue testing of materials. After that, PSALT was applied to test other products, e.g. capacitors (Starr & Endicolt, 1961), insulations (Solomon et al., 1976), and integrated circuits (Chan, 1990). Except for practical

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applications, statistical inference methods for PSALT data have also attracted many attentions in past decades (Yin & Sheng, 1987)– (Kumar Mahto et al., 2020). For example, (Lin & Fei, 1991) proposed a nonparametric approach to estimate the lifetime distribution under PSALT. (Mohie El-Din et al., 2017) utilized classical and Bayesian inference under PSALT when the lifetime of test units was assumed to follow a logistic exponential distribution.

The above studies on PSALT are all based on the assumption that the lifetime of test units is independent of each other. However, assets are assigned in a test stand for a certain stress acceleration scheme, and then the failures in the same chamber are no longer independent of each other. For example, (i) assets assembled by different sources of raw materials are assigned to the same test group, leading to the existence of batch effects (Seo & Pan, 2017); (ii) the stress is applied directly to the test stand instead of individual test assets, which will cause block effects in the resulting data (Kensler et al., 2014). It can be seen that both cases may result in a group structure, which means that the lifetimes within the same group may be correlated. If these correlations are ignored in data analysis, inaccurate results will be obtained. Moreover, with the limitation of testing time, the assets will be periodically removed at some certain time points during the testing process, and thus the collected data is progressively censored (Wang et al., 2014). Therefore, the data with group effects and progressively censored are common in PSALT.

1.2. Motivation

The research to be proposed is motivated by a real example from the insulating oil test (Nelson, 2009). The experimenter increased voltage linearly with time at a specified rate, v_i(V/s), in order to make specimens down faster, and the breakdown voltage was recorded. The dataset can be found in Table 1, which includes three groups, each containing 60 products and the failures are observed.

Due to the diversity of raw materials and stress profiles, group effects may exist in this experiment. To illustrate this situation, assume that the lifetime of unit in each group independently follows Weibull distribution, and that the relationship between scale parameter and the stress satisfies the inverse power law. We pool all the data together and estimate the model parameters by the maximum likelihood (ML) method. Then, the residuals between fitted cumulative distribution function (CDF) and empirical CDF are calculated. Figure 1 is a violin chart of the fitted residuals for each group. It can be intuitively seen that the differences among three groups are significant. For illustrating this point clearly, we perform analysis of variance (ANOVA) based on the residuals. Because the variance of each group is not homogeneous, we use Kruskal-Wallis (KW) test by ranks that is a nonparametric method to implement one-way ANOVA test (Kruskal & Wallis, 1952). The test statistic that reflects the variance in ranks is KW chi-squared statistic, which asymptotically follows the chi-square distribution under the null hypothesis that the medians of all groups are equal. More details can be found in (Kruskal & Wallis, 1952). Using the function *kruskal.test()* in *R* software, ANOVA for the residuals can be implemented easily. The value of KW chi-squared

Table 1. The PSALT data of insulating oils (Nelson, 2009).

Groups	v_i(V/s)	Breakdown Voltage(V)
1	10	34 34 34 35 35 35 36 38 38 38 38 39 39 39 40 40 40 40 41 41 41 41 41 41 42 42 42 42 42 43 43 43 43 43 44 44 44 44 44 44 44 45 45 46 46 46 46 46 47 47 47 47 47 48 49 49 49 50 51 52
2	100	34 36 37 39 42 43 43 43 44 45 45 45 45 45 46 46 46 46 46 47 47 47 47 48 48 48 49 49 49 49 49 50 50 50 50 50 50 51 51 52 52 52 52 52 53 53 53 53 53 53 53 53 53 54 54 54 55 55 55 58
3	1000	41 41 51 51 51 51 53 53 53 53 54 54 54 54 55 55 55 55 56 56 56 56 57 57 57 57 57 57 57 58 58 58 59 59 59 59 60 60 60 60 60 60 61 61 62 62 62 62 63 63 63 63 63 64 64 65 65 65 65 69

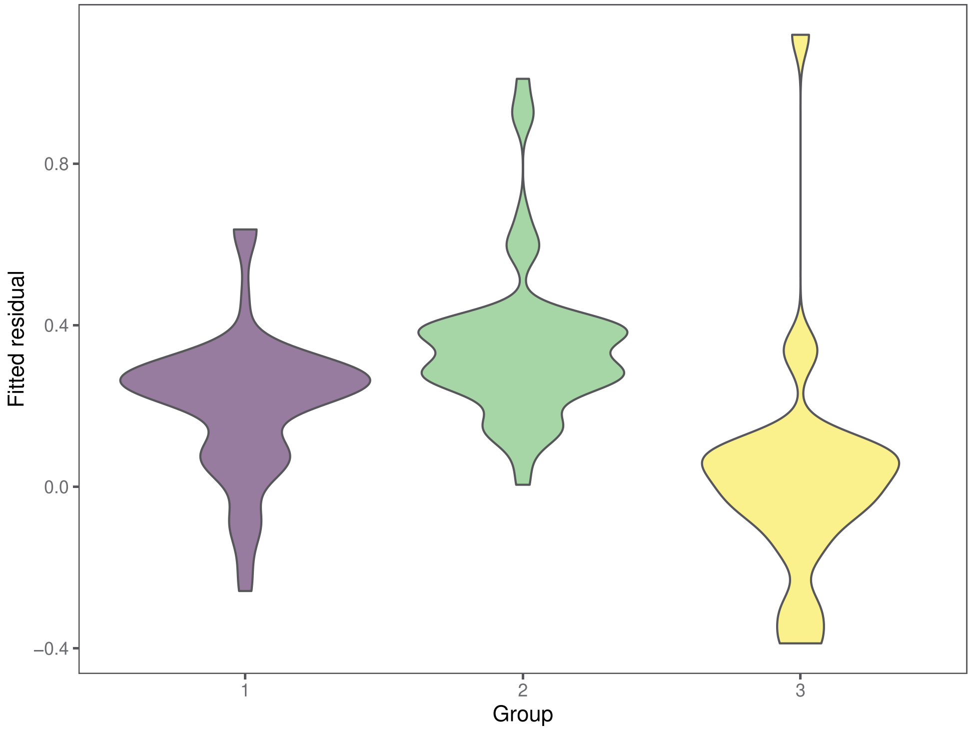

Figure 1. The residuals of each group.

statistic is equal to 72.041, and thus the p-value is 2.272×10^{-16} which is less than the significance level 0.05. Thus, we accordingly reject the null hypothesis and can conclude that there are significant differences between the test groups. That is, group effects exist in the data and should be included in the model. Otherwise, it may cause large bias in estimating the reliability of asset and results in wrong decisions (Seo & Pan, 2017; Zhuang et al., 2021).

Based on the observed patterns of the data, our goal is to solve the following two problems: (1) How to build a model for the PSALT data under progressive censoring with group effects? (2) How to extrapolate the characteristics lifetime under usual operating conditions? As we can see, whether the second problem can be solved depends on the first problem, and the second one is a general issue concerned by the manufacturer, which will help make a series of reliability decisions, to name a few, warranty policy, inventory control, design of new product, and so on.

1.3. Related work

Traditional ways to analyze product reliability often assume that the data come from a randomly designed experiment (Meeker & Escobar, 1998; Zhang et al., 2022). Nonetheless, when external effects change (e.g. block or batch effects), the data may no longer be completely randomized, instead they usually lead to grouped structures of experimental units. (León et al., 2009) claimed that if these external effects are ignored in the model, it would cause unreasonable estimates of quantile lifetime and probabilities of failure at the usual stress level, as well as misleading predictions of the failure time for a new unit. Other studies have also emphasized the necessity to incorporate external effects into lifetime analysis (Feiveson & Kulkarni, 2000; Lv et al., 2017).

Several studies have been conducted to incorporate group effects into analysis. For usual stress test (UST), (Zhuang et al., 2021) considered both heavy censoring and batch effects in the model. And they found that ignoring the group effects in the interval failure data will cause inaccurate predicted number of failures. For constant stress ALT (CSALT), (Freeman & Vining, 2010) provided two-stage (TS) method to analyze data from designed experiments which contain subsampling. However, they only considered the point estimation of the model parameters, and the estimate bias cannot be neglected in the case of small sample size. More importantly, the interval estimation cannot be obtained by the TS method, which is more useful in describing uncertainty of parameters. Therefore, the TS method has been extended in recent years by (Wang et al., 2016, 2019; Lv et al., 2019). For example, (Wang et al., 2016) developed a bootstrap method based on an unbiased factor, which could correct estimate bias and obtain interval estimation simultaneously. For step stress ALT (SSALT), (Seo & Pan, 2017) proposed a generalized linear mixed model to take the group effects into account under exponential distribution. They used adaptive Gaussian quadrature and integrated nested Laplace approximation to estimate the model parameters. (Wang, 2020) extended the model of (Seo & Pan, 2017) under the assumption of Weibull distribution.

The above works primarily deal with analyzing UST, CSALT and SSALT data for reliability experiments with group effects. However, the case of PSALT has not received much attention in the literature. In addition, they only considered conventional censoring schemes such as type-I censoring or type-II censoring, while progressive censoring allows the removal of test units at non-terminal points, and utilizes the available resources effectively, which is more flexible and efficient than conventional censoring schemes (Balakrishnan & Aggarwala, 2000; Montanari & Cacciari, 1988). To fill this gap, in this article, we first construct a model for the PSALT data with group effects by introducing random variables into the scale parameter. To the best of our knowledge, it has been not well studied yet. Second, we incorporate progressive censoring scheme into the PSALT. According to this generalized censoring scheme, engineers can carry out more flexible experiment strategies in the design stage. Then, the TS and Gauss-Hermite (GH) quadrature methods are proposed to obtain the point estimates as well as interval estimates of the model parameters.

1.4. Overview

The rest of the paper is organized as follows. Section 2 introduces the modeling framework for PSALT with group effects under progressive censoring. Section 3 considers the statistical inference for the proposed model based on two different methods. Section 4 is devoted to simulation studies, in which the results of neglecting group effects are assessed under different scenarios. A case study is provided to illustrate the performance of the proposed model in Section 5. Finally, we give some conclusions and discussions of this paper.

2. Model

2.1. PSALT model with group effects

Let T be the lifetime of an asset and assume that T follows the Weibull distribution with scale and shape parameters a and b, respectively. The probability density function (PDF) and CDF of T are:

$$f(t) = \frac{bt^{b-1}}{a^b}\exp\left\{-\left(\frac{t}{a}\right)^b\right\}\text{and } F(t) = 1 - \exp\left\{-\left(\frac{t}{a}\right)^b\right\}, \quad a, b > 0. \tag{1}$$

In PSALT, the scale parameter a is often assumed to have a relationship with the function of stress. In this paper, we assume the relationship satisfies the inverse power law, i.e.

$$a(t) = \frac{1}{c[s(t)]^d}, \tag{2}$$

where c and d are the unknown parameters that are positive, and $s(t)$ is the stress level, which is a function of t. Let s_0 be the used stress level, and the corresponding characteristic lifetime is $a_0 = (cs_0{}^d)^{-1}$. Furthermore, the cumulative exposure model (Nelson, 2009) is assumed in this paper, which means the distribution of the remaining life of a test asset depends only on the cumulative exposure it has received, no matter how it was exposed. For Weibull distribution, the shape parameters are empirically found to be correlated with the failure mechanism. Thus, we assume that $b_0 = b_1 = \cdots = b_k = b$ to guarantee that the failure mechanism under PSALT remains unchanged. See (Nelson, 2009) for details. Let $s_i(t) = v_i t, i = 1, \ldots, k$, where v_i is the increasing rate of stress in the i-th group which is a constant given by engineers before the experiment. Note that the linear form of $s_i(t)$ is common in practical experiments and has been widely utilized in literature (Abdel-Hamid & Al-Hussaini, 2011; Mohie El-Din et al., 2017). Except for linear increasing stress, there are also other forms of varying stress in PSALT, such as: cyclical stress (Cheng & Elsayed, 2017; Zhu et al., 2021; Kim & Sung, 2022), randomly varying stress (Gerville-Reache & Nikulin, 2007; Zheng & Ellingwood, 1998). Furthermore, we introduce random variables μ_is to reflect group effects and assume that group effects are exerted to the scale parameter a_i, which is given by

$$-\log(a_i(t, \mu_i)) = \log c + d\log(v_i t) + \log \mu_i, \quad i = 1, \ldots, k, \tag{3}$$

where $\log \mu_i$ is assumed to follow normal distribution:

$$\log \mu_i \sim \mathbb{N}(0, \sigma^2), \quad i = 1, \ldots, k. \tag{4}$$

σ^2 is a variance component of the group effects and it is one of the unknown parameters in the model. Under the cumulative exposure model, the CDF under $s_i(t)$ given μ_i for an asset in the i-th group is: $G_i(t|\mu_i) = F_i(\Delta t|\mu_i), i = 1, \ldots, k,$ where $\Delta t|\mu_i = \int_0^t \frac{\mathrm{d}w}{1/\left[\mu_i c v_i^d w^d\right]} = \frac{\mu_i c v_i^d t^{d+1}}{d+1}$, and $F_i(\cdot)$ is the CDF defined by (1). Therefore, given μ_i, the PDF and CDF can be formulated as

$$g_i(t|\mu_i) = \frac{\lambda}{\alpha_i}\left(\frac{t}{\alpha_i}\right)^{\lambda-1} \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\lambda}\right] \text{and } G_i(t|\mu_i) = 1 - \exp\left[-\left(\frac{t}{\alpha_i}\right)^{\lambda}\right], \tag{5}$$

respectively, where

$$\alpha_i = \left(\frac{d+1}{\mu_i c v_i^d}\right)^{1/(d+1)} \quad \text{and} \quad \lambda = b(d+1). \tag{6}$$

2.2. Progressive censoring

Progressive censoring is widely used in the reliability experiment. (Herd, 1956) was the first to discuss estimation of the population parameters based on progressively censored samples. (Cohen, 1963) discussed the importance of progressive censoring in reliability experiments. Due to its effectiveness for saving experimental time, many scholars have incorporated progressive censoring scheme into reliability analysis in recent years (Chen et al., 2016; Wang et al., 2014; Singh et al., 2022; Mahto et al., 2022). There are two types of progressive censoring schemes, called type-I and type-II, which are introduced detailedly as follows:

(1) **Type-I progressive censoring**: In the i-th group, the experimenter conducts a life test at each fixed time $(T_{i1}, \ldots, T_{im_i})$ until time to T_{im_i}, records the number of failures c_{ij}, and randomly remove R_{ij} non-failed units at each fixed time.

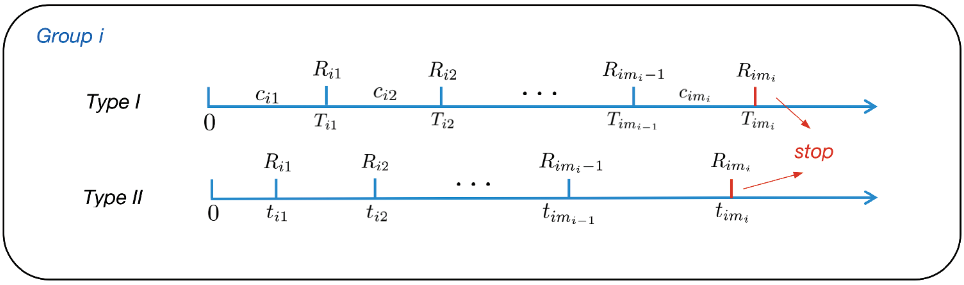

Figure 2. Two types of progressive censoring for the i-th group.

(2) **Type-II progressive censoring**: In the i-th group, the experimenter records the first m_i failures: $(t_{i1}, \dots, t_{im_i})$. At the same time, when one unit fails, R_{ij} non-failed units will be randomly removed.

Figure 2 shows the structure of the two censoring schemes for the i-th group. It is clear that there is a fundamental difference between the two schemes. In the case of type-I censoring, the duration of test is fixed and the number of failures is random, while in the case of type-II censoring, the duration of test is random and the number of failures is fixed. Without loss of generality, we mainly consider the case of type-II progressive censoring scheme, because the procedures of statistical inference are similar for the two cases.

Assume that there are k groups in the PSALT with progressive type-II censoring scheme. In the i-th group, a number of n_i identical units are tested. And we suppose that the failure number is m_i and the progressive censoring scheme is $R_i = (R_{i1}, \dots, R_{im_i})$, where $R_{ij} \geq 0$ and $\sum_{j=1}^{m_i} R_{ij} + m_i = n_i$. Let $t_{i1} < \dots < t_{im_i}$ be the observed failures in the i-th group. At the j-th failure time t_{ij}, R_{ij} units are removed, $j = 1, \dots, m_i$. Specially, when $R_{i1} = R_{i2} = \dots = R_{i(m-1)} = 0$, $R_{im_i} = n_i - m_i$, which corresponds to the conventional type-II right censoring scheme. Thus, the observed data is $\mathcal{D} = \{(t_{ij}, R_{ij}), i = 1, \dots, k, j = 1, \dots, m_i\}$.

3. Inference

In this section, two methods for estimating model parameters are briefly discussed. The first one is to use the TS method to obtain the point estimation and utilize bootstrap method to calculate the corresponding interval estimation. The second one is to carry out GH quadrature to approximate the likelihood function and the interval estimation of parameters can be computed by asymptotic normality theorem.

3.1. Two-stage method

The procedures of the direct TS approach with application to PSALT under progressive censoring with group effects are described as follows:

1. The goal of the first stage is to obtain the estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\alpha_1, \dots, \alpha_k, \lambda)$. Given the observed data $\mathcal{D}$, the likelihood function is:

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{k} \left[A_i \prod_{j=1}^{m_i} g_i(t_{ij}|\mu_i) \left(1 - G_i(t_{ij}|\mu_i)\right)^{R_{ij}} \right], \tag{7}$$

where $A_i = n_i \prod_{s=1}^{m_i-1} \left(n_i - s - \sum_{q=1}^{s} R_{iq}\right)$. Then, the log-likelihood function can be formulated as follows:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{k}\left\{\log A_i + \sum_{j=1}^{m_i}\log\left[g_i\left(t_{ij}|\mu_i\right)\left(1-G_i\left(t_{ij}|\mu_i\right)\right)^{R_{ij}}\right]\right\}, \tag{8}$$

where $g_i\left(t_{ij}|\mu_i\right)$ and $g_i\left(t_{ij}|\mu_i\right)$ are expressed by (5). Taking the first partial derivatives of log-likelihood function in (8) with respect to $\boldsymbol{\theta}$ and equating each to zero, we obtain the following equations:

$$\frac{\partial\ell}{\partial\alpha_i} = \sum_{j=1}^{m_i}\left[\frac{\lambda}{\alpha_i} - (R_{ij}+1)\left(\frac{\lambda t_{ij}^{\lambda}}{\alpha_i^{\lambda+1}}\right)\right] = 0, \quad i = 1,\ldots,k, \tag{9}$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left\{\frac{1}{\lambda} + \log\left(\frac{t_{ij}}{\alpha_i}\right)\left[1-\left(R_{ij}+1\right)\left(\frac{t_{ij}}{\alpha_i}\right)^{\lambda}\right]\right\} = 0. \tag{10}$$

Given λ, the solution of (9) is

$$\alpha_i = \left(\frac{\sum_{j=1}^{m_i}(R_{ij}+1)t_{ij}^{\lambda}}{m_i}\right)^{1/\lambda}, \quad i = 1,\ldots,k. \tag{11}$$

Then, substituting (11) into (10), after some algebraic calculations, we get

$$\frac{M}{\lambda} + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[1-\frac{(R_{ij}+1)m_i t_{ij}^{\lambda}}{\sum_{j=1}^{m_i}(R_{ij}+1)t_{ij}^{\lambda}}\right]\log\left(\frac{t_{ij}m_i^{\frac{1}{\lambda}}}{\left(\sum_{j=1}^{m_i}(R_{ij}+1)t_{ij}^{\lambda}\right)^{\frac{1}{\lambda}}}\right) = 0, \tag{12}$$

where $M = \sum_{i=1}^{k} m_i$ is the number of all failure units. The MLE of λ is the solution of (12) that can be computed using some iteration procedure, e.g. the Newton-Raphson iterative algorithm and the quasi-Newton method, which is denoted as $\hat{\lambda}$. The existence of the solution of (12) is shown in Appendix. Replacing λ by $\hat{\lambda}$ in (11), the MLE of α_i can be obtained and denote it as $\hat{\alpha}_i$.

2. The second stage aims to obtain the parameters in distribution of group effects. We take logarithm for (6), where α_is is replaced by $\hat{\alpha}_i$s. In this stage, $\hat{\alpha}_i$s obtained in the first stage are treated as the 'obsevations' and let $y_i = \log\hat{\alpha}_i$. Then we have the following linear model:

$$y_i = \log\hat{\alpha}_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \tag{13}$$

where $x_i = \log v_i, \beta_0 = \frac{\log[(d+1)/c]}{d+1}, \beta_1 = -\frac{d}{d+1}$ and $\varepsilon_i = -\frac{\log\mu_i}{d+1}$. This is a typical linear model, where β_0 and β_1 are the intercept parameter and the slope parameter, respectively. The error terms ε_is are independent of each other for different groups, and follow normal distribution $N(0,\sigma_\varepsilon^2)$, where $\sigma_\varepsilon^2 = \frac{\sigma^2}{(d+1)^2}$ is the variance of ε_i. According to the Gauss-Markov theorem (Rao, 1965), the estimators of β_1 and β_0 are, respectively, given by

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{k}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{k}(x_i-\bar{x})^2} \text{and} \hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}, \tag{14}$$

where $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$ and $\bar{y} = \frac{1}{k}\sum_{i=1}^{k} y_i$. Then, using following formula to get the estimates of parameters $(\hat{b},\hat{c},\hat{d})$.

$$\begin{aligned} \hat{b} &= \hat{\lambda}\left(\hat{\beta}_1 + 1\right), \\ \hat{c} &= \frac{1}{\left(\hat{\beta}_1+1\right)\exp\left\{\frac{\hat{\beta}_0}{\hat{\beta}_1+1}\right\}}, \\ \hat{d} &= -\frac{\hat{\beta}_1}{\hat{\beta}_1+1}. \end{aligned} \tag{15}$$

Based on $\hat{\beta}_0$ and $\hat{\beta}_1$, we can get the residuals $\hat{\delta}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$, $i = 1, 2, \ldots, k$. Thus, the estimate of σ^2 can be obtained by

$$\hat{\sigma}^2 = \frac{\left(\hat{d}+1\right)^2 \sum_{i=1}^{k} \hat{\delta}_i^2}{k-1}. \tag{16}$$

By plug-in method, for any continuous function of the model parameters, e.g. $C(b, c, d, \sigma^2)$, the estimate could be $C\left(\hat{b}, \hat{c}, \hat{d}, \hat{\sigma}^2\right)$. Specially, for the characteristic lifetime at the normal used condition $a_0 = \left(c{s_0}^d\right)^{-1}$, the estimate can be obtained by

$$a_0 = \left(\hat{c}{s_0}^{\hat{d}}\right)^{-1}. \tag{17}$$

Using the delta method, the interval estimate of certain continuous function of the parameters $\{\alpha_i, \lambda, i = 1, \cdots, k\}$ can easily be calculated (Zhuang et al., 2021). However, the estimate of σ^2 is based on 'pseudo sample' $\hat{\alpha}_i$s, which implies that the interval estimate of σ^2 cannot be calculated directly by the asymptotic normality theorem (Freeman & Vining, 2010). To solve this problem, we used bootstrap method to construct the interval estimates of the model parameters. This method has been widely used in reliability field, for example (Bera & Jana, 2022; Palayangoda & Ng, 2021). The procedure of the bootstrap resampling approach is provided in the Algorithm 1.

Algorithm 1: Bootstrap algorithm based on TS method.

Input: Observation data $\mathcal{D}$.

Output: Point estimates and corresponding interval estimates for parameters $\hat{\vartheta} = (\hat{b}, \hat{c}, \hat{d}, \hat{\sigma}, \hat{a}_0)$.

1 Obtain the estimate $\hat{\boldsymbol{\theta}}$ based on the first stage;
2 **for** b *in* $\{1, 2, \ldots, B\}$
3 Generate bootstrap sample $\mathcal{D}^*$ from (5), when the parameter vector $\boldsymbol{\theta}$ is replaced by $\hat{\boldsymbol{\theta}}$;
4 Caculate $\hat{b}, \hat{c}, \hat{d}, \hat{\sigma}, \hat{a}_0$, by Equations (15), (16) and (17), respectively;
5 **end**
6 Calculate the point and interval estimates of the parameters based on the bootstrap results in steps 3 and 4.

Remark: There are several ways to implement bootstrap approach. Another bootstrap approach is a little different in generating bootstrap samples compared with Algorithm 1. Firstly, we generate B random numbers $\log \mu_i^{(B)}$ from $N(0, \hat{\sigma}^2)$, $i = 1, \ldots, k$ based on the model (4). Then, the bootstrap sample in each group from $F\left(t; \mu_i^{(B)}, \hat{\sigma}\right)$ in (5) can be generated. Other steps are the same as Algorithm 1. Since the results obtained by the two bootstrap methods are similar in the simulation studies, we only use Algorithm 1 to obtain the interval estimation of the model parameters.

3.2. Gauss – Hermite quadrature

GH quadrature is a form of Gaussian quadrature for approximating intractable integrals. Compared with Monte–Carlo integration, GH quadrature could provide an accurate approximation for the integral with a much lower computational budget. (Seo & Pan, 2017) and (Wang, 2020) applied this method into SSALT with group effects. The GH quadrature applied in PSALT with group effects can be summarized as follows:

The marginal likelihood of all of the observations in all groups can be constructed by integrating out the group effect for each group and then multiplying the likelihoods of all groups. That is,

$$L(\xi) = \prod_{i=1}^{k} A_i \int_{-\infty}^{\infty} \left[\prod_{j=1}^{m_i} g_i\left(t_{ij}|\mu_i\right)\left(1 - G_i\left(t_{ij}|\mu_i\right)\right)^{R_{ij}} \right] \pi\left(\mu_i\right) \mathrm{d}\mu_i, \tag{18}$$

where $\xi = (b, c, d, \sigma)$ is the vector that needs to be estimated, $\pi(\mu_i)$ is the PDF of μ_i, which follows log-normal distribution, $\pi(\mu_i) = (2\pi\sigma^2\mu_i^2)^{-1/2} \exp\{-\log^2 \mu_i/(2\sigma^2)\}$. Because the intractable integration in the likelihood could not be solved in closed form, we use GH quadrature to achieve an accurate approximation of log-likelihood in (18). It is important to note that, in order to use GH quadrature, a term with the form e^{-x^2} should exist in integral. Let $\mu_i = \exp(\sqrt{2}\sigma\tau_i)$, the log-likelihood in (18) can be expressed as

$$\begin{aligned} \ell(\xi) &= \sum_{i=1}^{k} \log\left\{ \int_{-\infty}^{\infty} \left[\prod_{j=1}^{m_i} g_i\left(t_{ij}|\tau_i\right)\left(1 - G_i\left(t_{ij}|\tau_i\right)\right)^{R_{ij}} \right] \times \frac{e^{-\tau_i^2}}{\sqrt{\pi}} \mathrm{d}\tau_i \right\} + \sum_{i=1}^{k} \log(A_i) \\ &\approx \sum_{i=1}^{k} \log\left\{ \sum_{h=1}^{l} \left[\prod_{j=1}^{m_i} g_i\left(t_{ij}|\tau_{ih}\right)\left(1 - G_i\left(t_{ij}|\tau_{ih}\right)\right)^{R_{ij}} \right] \times \frac{w_{ih}}{\sqrt{\pi}} \right\} + \sum_{i=1}^{k} \log(A_i), \end{aligned} \tag{19}$$

where l is the number of quadrature points, v_{ih}s are fixed evaluation points and w_{ih}s are the associated weights. The values of $\{(\tau_{ih}, w_{ih}), h = 1, \ldots, l\}$ are related with l, and the details can be found in (Liu & Pierce, 1994). Notice that a small value of l will cause inaccurate approximation, while a large value of l will increase the computational cost. Thus, we choose $l = 20$ as recommended by (Liu & Pierce, 1994). After approximating the log-likelihood in (19), the estimation of parameters is then achieved through numerical optimization algorithm, for instance, Newton-Raphson algorithm. However, the result for direct optimizing $\ell(\xi)$ is sensitive to the initial values of the parameters. The point estimate by the TS method is used as the initial value in this paper.

4. Simulation study

In this section, simulation studies are implemented to assess the proposed model and inference methods. We choose the number of group $k = 3, 5$ and 7. For each k, five progressive censoring schemes (progressive or conventional type-II censoring) are considered with different sample sizes $n_i,\ i = 1, \ldots, k$. The details of the sample sizes and censoring schemes are summarized in Table 2. For each scenario, the data are generated from the Weibull distribution as specified in (5) with the model parameters $(b, c, d) = (1.5, 0.5, 2)$. Thus, the values of β_0 and β_1 are equal to 0.597 and −0.667, respectively. Let the usual stress level be $s_0 = 1$ and thus the corresponding characteristic lifetime is $a_0 = 2$. In addition, variance component of group effects σ is assigned as 0, 0.3, 0.5 and 0.8. $\sigma = 0$ implies that there are no group effects in the model, for which case we want to show the performance or robustness of the proposed model when the model is misspecified. Under these settings, we will generate 1000 samples for each combination of parameters and censoring schemes.

In order to illustrate the necessity for considering group effects in PSALT data, we also add two other models for comparison. The first model assumes that all test units are randomly independent and the correlations of observations among groups are ignored, which corresponds to $\sigma = 0$ in the

Table 2. The progressive censoring schemes for $k = 3, 5$ and 7.

	$n_1, \ldots, n_k$	$v_1, \ldots, v_k$	$r_1, \ldots, r_k$	$R_1, \ldots, R_k$
1	(20, 15, 10)	(0.2, 0.3, 0.4)	(12, 9, 6)	$R_1 = (0, \ldots, 0, 8)$ $R_2 = (0, \ldots, 0, 6)$ $R_2 = (0, \ldots, 0, 4)$
2	(20, 15, 10)	(0.2, 0.3, 0.4)	(12, 9, 6)	$R_1 = (8, 0, \ldots, 0)$ $R_2 = (4, 0, \ldots, 0)$ $R_3 = (4, 0, \ldots, 0)$
3	(20, 15, 10)	(0.2, 0.3, 0.4)	(12, 9, 6)	$R_1 = (4, 0, \ldots, 0, 4)$ $R_2 = (2, 0, \ldots, 0, 2)$ $R_3 = (2, 0, \ldots, 0, 2)$
4	(20, 15, 10)	(0.2, 0.3, 0.4)	(12, 9, 6)	$R_1 = (1, \ldots, 1, 0, \ldots, 0)$ $R_2 = (1, \ldots, 1, 0, \ldots, 0)$ $R_3 = (1, \ldots, 1, 0, \ldots, 0)$
5	(20, 15, 10)	(0.2, 0.3, 0.4)	(12, 9, 6)	$R_1 = (0, \ldots, 0, 1, \ldots, 1)$ $R_2 = (0, \ldots, 0, 1, \ldots, 1)$ $R_3 = (0, \ldots, 0, 1, \ldots, 1)$
6	(30, 25, 20, 15, 10)	(0.2, …, 0.6)	(18, 15, 12, 9, 6)	$R_1 = (0, \ldots, 0, 12)$ $R_5 = (0, \ldots, 0, 4)$
7	(30, 25, 20, 15, 10)	(0.2, …, 0.6)	(18, 15, 12, 9, 6)	$R_1 = (12, 0, \ldots, 0)$ $R_5 = (4, 0, \ldots, 0)$
8	(30, 25, 20, 15, 10)	(0.2, …, 0.6)	(18, 15, 12, 9, 6)	$R_1 = (6, 0, \ldots, 0, 6)$ $R_5 = (2, 0, \ldots, 0, 2)$
9	(30, 25, 20, 15, 10)	(0.2, …, 0.6)	(18, 15, 12, 9, 6)	$R_1 = (1, \ldots, 1, 0, \ldots, 0)$ $R_5 = (1, \ldots, 1, 0, \ldots, 0)$
10	(30, 25, 20, 15, 10)	(0.2, …, 0.6)	(18, 15, 12, 9, 6)	$R_1 = (0, \ldots, 0, 1, \ldots, 1)$ $R_5 = (0, \ldots, 0, 1, \ldots, 1)$
11	(40, 35, …, 15, 10)	(0.2, …, 0.8)	(24, 21, …, 9, 6)	$R_1 = (0, \ldots, 0, 16)$ $R_7 = (0, \ldots, 0, 4)$
12	(40, 35, …, 15, 10)	(0.2, …, 0.8)	(24, 21, …, 9, 6)	$R_1 = (16, 0, \ldots, 0)$ $R_7 = (4, 0, \ldots, 0)$
13	(40, 35, …, 15, 10)	(0.2, …, 0.8)	(24, 21, …, 9, 6)	$R_1 = (8, 0, \ldots, 0, 8)$ $R_7 = (2, 0, \ldots, 0, 2)$
14	(40, 35, …, 15, 10)	(0.2, …, 0.8)	(24, 21, …, 9, 6)	$R_1 = (1, \ldots, 1, 0, \ldots, 0)$ $R_7 = (1, \ldots, 1, 0, \ldots, 0)$
15	(40, 35, …, 15, 10)	(0.2, …, 0.8)	(24, 21, …, 9, 6)	$R_1 = (0, \ldots, 0, 1, \ldots, 1)$ $R_7 = (0, \ldots, 0, 1, \ldots, 1)$

proposed model. The other model assumes that all groups have a common shape parameter, while the scale parameters of each group, $a_i(t) = \frac{1}{c(v_i t)^d}$, are different. The two models are denoted as 'pooled model' and 'fixed model', respectively. For the pooled model, the parameter estimation will be obtained by ML method and the corresponding interval estimation can be calculated by the asymptotic normality theorem. For the fixed model, the parameters are estimated by TS approach. Thus, for a generated sample, we will fit the data by the three models, and for the proposed model, both TS and GH methods are utilized to estimate model parameters.

Tables 3–6 list the relative bias (RB) and the root-mean-squared error (RMSE) of parameter estimators $\hat{\vartheta} = (\hat{b}, \hat{c}, \hat{d}, \hat{\sigma}, \hat{a}_0)$ based on the three models under different simulation scenarios, where 'GH', 'TS', 'Pooled' and 'Fixed' denote the results based on the proposed model with GH and TS methods, pooled model, and fixed model, respectively. For the estimates obtained from 1000 samples, we find that there are many abnormal estimates based on the pooled and fixed models, especially in the case of large σ. To make the results comparable, we remove the 10% upper and the 10% lower of 1000 estimates in each model. The RB and RMSE for the rest 800 estimates are defined as follows:

$$\mathrm{RB} = \frac{1}{800}\sum_{i=1}^{800}\frac{\hat{\vartheta}_i - \vartheta}{\vartheta}, \quad \mathrm{RMSE} = \sqrt{\frac{1}{800}\sum_{i=1}^{800}\left(\hat{\vartheta}_i - \vartheta\right)^2}. \tag{20}$$

Table 3. RB and RMSE based on different models under $\sigma = 0$.

	RB															
	GH				TS				Pooled				Fixed			
	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0
1	0.098	−0.004	0.108	0.018	0.102	0.022	0.175	−0.006	0.098	−0.009	0.166	0.042	0.168	0.027	0.224	−0.018
2	0.047	−0.010	0.133	0.021	0.031	−0.002	0.221	0.013	0.043	−0.011	0.213	0.044	0.092	0.013	0.224	−0.001
3	0.094	−0.020	0.057	0.028	0.081	−0.004	0.115	0.009	0.087	−0.026	0.131	0.055	0.149	0.011	0.147	−0.007
4	0.054	−0.015	0.127	0.026	0.051	0.001	0.182	0.011	0.056	−0.018	0.191	0.049	0.114	0.019	0.214	−0.008
5	0.073	−0.006	0.071	0.004	0.073	0.016	0.118	−0.019	0.072	−0.011	0.185	0.042	0.141	0.031	0.177	−0.031
6	0.036	0.008	0.049	−0.002	0.051	0.040	0.099	−0.032	0.034	0.004	0.044	0.001	0.125	0.050	0.150	−0.041
7	0.031	−0.010	0.023	0.016	0.031	0.005	0.069	0.001	0.030	−0.013	0.023	0.018	0.089	0.017	0.080	−0.010
8	0.034	0.001	0.044	0.006	0.040	0.024	0.098	−0.017	0.034	−0.003	0.036	0.009	0.114	0.036	0.124	−0.028
9	0.015	0.005	0.046	0.001	0.019	0.025	0.088	−0.018	0.016	0.002	0.041	0.004	0.085	0.040	0.112	−0.033
10	0.026	−0.001	0.051	0.007	0.031	0.026	0.107	−0.02	0.024	−0.004	0.046	0.010	0.105	0.040	0.146	−0.032
11	0.033	0.006	0.009	−0.003	0.046	0.032	0.051	−0.029	0.029	0.003	0.008	−0.001	0.121	0.039	0.085	−0.035
12	0.015	−0.006	0.017	0.008	0.015	0.007	0.059	−0.004	0.014	−0.007	0.015	0.009	0.077	0.016	0.064	−0.013
13	0.022	0.001	0.025	0.002	0.029	0.021	0.070	−0.018	0.021	−0.002	0.021	0.004	0.097	0.029	0.09	−0.025
14	0.020	0.005	0.008	−0.003	0.024	0.021	0.043	−0.019	0.019	0.003	0.005	−0.001	0.081	0.032	0.062	−0.029
15	0.017	0.006	0.023	−0.004	0.026	0.028	0.060	−0.025	0.015	0.003	0.021	0.001	0.096	0.039	0.093	−0.035
	RMSE															
1	0.701	0.116	1.455	0.473	0.736	0.126	1.684	0.483	0.695	0.114	1.437	0.472	0.850	0.145	1.989	0.521
2	0.642	0.115	1.436	0.471	0.694	0.125	1.789	0.492	0.645	0.115	1.462	0.473	0.758	0.133	1.802	0.499
3	0.663	0.105	1.211	0.441	0.698	0.115	1.473	0.456	0.670	0.106	1.262	0.445	0.777	0.119	1.545	0.458
4	0.687	0.116	1.494	0.472	0.715	0.125	1.640	0.480	0.679	0.113	1.431	0.469	0.793	0.134	1.812	0.486
5	0.664	0.120	1.426	0.475	0.697	0.132	1.623	0.489	0.665	0.117	1.431	0.472	0.792	0.146	1.864	0.509
6	0.347	0.049	0.604	0.196	0.377	0.055	0.684	0.210	0.344	0.048	0.594	0.193	0.449	0.061	0.767	0.221
7	0.317	0.046	0.577	0.187	0.345	0.048	0.642	0.191	0.315	0.046	0.576	0.187	0.395	0.050	0.665	0.195
8	0.332	0.049	0.599	0.197	0.367	0.053	0.686	0.207	0.329	0.048	0.587	0.195	0.431	0.056	0.721	0.211
9	0.320	0.047	0.583	0.192	0.348	0.051	0.654	0.204	0.318	0.047	0.575	0.189	0.401	0.055	0.684	0.207
10	0.330	0.047	0.580	0.191	0.371	0.051	0.683	0.198	0.326	0.047	0.572	0.190	0.430	0.055	0.752	0.208
11	0.233	0.031	0.378	0.125	0.269	0.036	0.443	0.139	0.227	0.031	0.373	0.124	0.331	0.038	0.490	0.143
12	0.206	0.031	0.382	0.124	0.232	0.032	0.440	0.129	0.203	0.031	0.377	0.124	0.282	0.033	0.449	0.131
13	0.220	0.031	0.374	0.125	0.245	0.034	0.435	0.133	0.217	0.031	0.369	0.125	0.292	0.036	0.456	0.137
14	0.196	0.028	0.345	0.112	0.218	0.032	0.396	0.123	0.196	0.028	0.346	0.112	0.261	0.034	0.408	0.130
15	0.203	0.030	0.357	0.120	0.233	0.035	0.415	0.133	0.200	0.030	0.353	0.120	0.286	0.038	0.457	0.141

Table 4. RB and RMSE based on different models under $\sigma = 0.3$.

	RB															
	GH				TS				Pooled				Fixed			
	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0
1	0.087	−0.284	−0.197	0.074	0.092	−0.272	−0.135	0.032	0.098	−0.204	0.785	0.689	0.171	−0.273	−0.093	0.016
2	0.015	−0.291	−0.137	0.053	−0.018	−0.331	−0.076	0.021	0.014	−0.203	1.197	2.120	0.051	−0.311	−0.043	0.010
3	0.074	−0.185	−0.052	0.077	0.062	−0.193	0.027	0.039	0.072	−0.187	0.791	0.641	0.126	−0.216	0.012	0.008
4	0.046	−0.260	−0.142	0.049	0.028	−0.294	−0.124	0.008	0.052	−0.193	1.105	1.396	0.090	−0.310	−0.125	−0.011
5	0.087	−0.184	−0.072	0.108	0.076	−0.172	0.015	0.066	0.086	−0.190	0.802	0.544	0.138	−0.188	0.047	0.044
6	0.052	−0.061	0.057	0.078	0.078	−0.020	0.122	0.032	0.019	−0.105	0.053	0.141	0.158	−0.013	0.172	0.027
7	0.002	−0.066	0.123	0.079	−0.006	−0.045	0.219	0.057	−0.012	−0.091	0.125	0.122	0.052	−0.036	0.221	0.046
8	0.033	−0.063	0.086	0.079	0.044	−0.031	0.164	0.042	0.011	−0.098	0.064	0.130	0.115	−0.021	0.193	0.032
9	0.016	−0.051	0.095	0.066	0.018	−0.026	0.159	0.037	−0.003	−0.084	0.098	0.114	0.079	−0.013	0.182	0.024
10	0.018	−0.046	0.123	0.064	0.022	−0.009	0.221	0.025	0.003	−0.093	0.091	0.126	0.097	0.004	0.264	0.012
11	0.012	−0.035	0.058	0.046	0.037	0.002	0.113	0.010	−0.041	−0.084	0.027	0.103	0.109	0.006	0.144	0.004
12	0.007	−0.038	0.058	0.049	0.009	−0.022	0.115	0.032	−0.022	−0.068	0.024	0.084	0.067	−0.014	0.121	0.024
13	0.009	−0.039	0.067	0.052	0.031	−0.012	0.111	0.024	−0.035	−0.080	0.033	0.098	0.101	−0.005	0.134	0.017
14	0.005	−0.031	0.052	0.042	0.012	−0.008	0.097	0.019	−0.023	−0.066	0.025	0.083	0.073	0.003	0.116	0.008
15	0.033	−0.041	0.033	0.052	0.048	−0.010	0.078	0.025	−0.014	−0.085	0.009	0.105	0.123	−0.001	0.109	0.012
	RMSE															
1	1.136	0.489	3.587	1.007	1.191	0.540	3.937	1.001	1.005	0.212	5.610	6.182	1.338	0.566	4.271	1.019
2	1.024	0.501	3.815	0.973	1.060	0.611	4.588	1.007	0.955	0.217	6.906	22.885	1.164	0.605	4.704	1.015
3	1.086	0.337	3.323	0.986	1.142	0.386	3.802	1.002	0.975	0.217	4.948	4.919	1.242	0.467	3.992	1.002
4	1.075	0.466	3.629	1.002	1.114	0.558	3.897	1.017	0.97	0.225	6.990	12.409	1.212	0.595	4.113	1.026
5	1.087	0.266	3.213	0.985	1.129	0.310	3.742	0.993	0.987	0.212	5.416	3.738	1.244	0.376	4.044	1.012
6	0.566	0.092	1.144	0.409	0.609	0.096	1.276	0.388	0.553	0.098	1.122	0.469	0.704	0.103	1.365	0.396
7	0.568	0.092	1.283	0.409	0.594	0.096	1.503	0.408	0.558	0.094	1.245	0.443	0.642	0.099	1.519	0.406
8	0.564	0.091	1.216	0.404	0.598	0.094	1.371	0.391	0.549	0.094	1.150	0.447	0.665	0.097	1.445	0.392
9	0.531	0.090	1.096	0.394	0.563	0.095	1.257	0.394	0.531	0.094	1.109	0.433	0.621	0.098	1.307	0.393
10	0.548	0.094	1.186	0.404	0.583	0.103	1.369	0.399	0.543	0.098	1.138	0.456	0.658	0.105	1.481	0.406
11	0.394	0.064	0.741	0.268	0.423	0.066	0.817	0.262	0.384	0.072	0.753	0.332	0.483	0.068	0.871	0.267
12	0.383	0.063	0.739	0.269	0.395	0.065	0.804	0.267	0.376	0.067	0.738	0.303	0.439	0.065	0.813	0.269
13	0.389	0.066	0.761	0.282	0.406	0.067	0.802	0.273	0.384	0.071	0.756	0.325	0.463	0.069	0.826	0.276
14	0.364	0.065	0.707	0.275	0.377	0.068	0.749	0.275	0.366	0.069	0.722	0.311	0.416	0.071	0.771	0.274
15	0.405	0.066	0.766	0.282	0.421	0.067	0.808	0.271	0.403	0.073	0.781	0.338	0.483	0.071	0.854	0.272

Table 5. RB and RMSE based on different models under $\sigma = 0.5$.

	RB															
	GH				TS				Pooled				Fixed			
	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0
1	0.099	−0.722	−0.636	0.073	0.101	−0.715	−0.596	0.028	0.135	−0.398	1.853	34.415	0.179	−0.861	−0.706	−0.013
2	−0.012	−1.026	−0.766	0.002	−0.036	−0.986	−0.678	−0.018	0.054	−0.399	2.335	51.442	0.032	−0.983	−0.706	−0.042
3	0.045	−0.825	−0.623	0.039	0.049	−0.789	−0.542	0.018	0.092	−0.389	1.773	28.542	0.113	−0.836	−0.559	−0.011
4	0.036	−0.750	−0.637	0.017	0.015	−0.763	−0.581	−0.015	0.081	−0.375	2.311	51.510	0.073	−0.785	−0.595	−0.039
5	0.103	−0.670	−0.581	0.067	0.097	−0.667	−0.537	0.043	0.134	−0.376	1.904	27.874	0.158	−0.743	−0.599	0.011
6	0.083	−0.124	0.004	0.125	0.086	−0.115	0.094	0.096	−0.021	−0.238	0.205	0.415	0.171	−0.112	0.112	0.077
7	0.012	−0.137	0.073	0.115	−0.014	−0.152	0.190	0.112	−0.033	−0.212	0.348	0.386	0.045	−0.141	0.211	0.102
8	0.069	−0.119	0.033	0.124	0.055	−0.117	0.155	0.111	−0.016	−0.224	0.211	0.371	0.128	−0.112	0.169	0.096
9	0.032	−0.116	0.056	0.109	0.017	−0.118	0.146	0.094	−0.036	−0.204	0.284	0.348	0.079	−0.108	0.169	0.079
10	0.039	−0.103	0.113	0.103	0.014	−0.104	0.244	0.087	−0.037	−0.225	0.254	0.399	0.088	−0.097	0.292	0.071
11	0.033	−0.054	0.091	0.075	0.036	−0.046	0.166	0.062	−0.109	−0.195	0.069	0.284	0.109	−0.042	0.204	0.060
12	0.015	−0.058	0.101	0.073	0.009	−0.071	0.161	0.084	−0.061	−0.153	0.055	0.216	0.069	−0.062	0.165	0.074
13	0.025	−0.069	0.106	0.097	0.033	−0.063	0.159	0.084	−0.091	−0.182	0.073	0.263	0.104	−0.057	0.181	0.077
14	0.009	−0.064	0.116	0.095	0.009	−0.053	0.168	0.081	−0.071	−0.163	0.065	0.235	0.071	−0.044	0.189	0.071
15	0.050	−0.072	0.089	0.103	0.061	−0.062	0.134	0.088	−0.066	−0.197	0.042	0.291	0.137	−0.055	0.169	0.082
	RMSE															
1	1.638	0.914	4.007	1.487	1.708	0.948	4.430	1.498	1.259	0.281	11.222	324.712	1.898	1.106	4.720	1.484
2	1.479	1.313	4.608	1.453	1.517	1.277	4.789	1.468	1.211	0.292	13.265	463.323	1.651	1.236	4.641	1.445
3	1.559	1.152	4.441	1.494	1.611	1.045	4.735	1.494	1.217	0.287	11.213	322.868	1.751	1.152	4.923	1.491
4	1.538	0.942	4.004	1.444	1.571	1.013	4.298	1.447	1.203	0.289	13.146	513.553	1.720	1.035	4.379	1.457
5	1.555	0.958	3.986	1.453	1.608	0.924	4.370	1.464	1.263	0.283	11.327	280.123	1.766	1.044	4.434	1.479
6	0.838	0.166	2.167	0.772	0.888	0.178	2.782	0.791	0.756	0.172	2.169	1.277	1.007	0.187	2.986	0.809
7	0.826	0.167	2.541	0.794	0.858	0.183	3.241	0.841	0.772	0.168	2.622	1.364	0.919	0.186	3.238	0.826
8	0.820	0.158	2.166	0.749	0.867	0.174	2.845	0.821	0.751	0.162	2.262	1.104	0.952	0.179	3.002	0.807
9	0.784	0.156	2.045	0.738	0.826	0.173	2.638	0.794	0.751	0.160	2.263	1.122	0.902	0.178	2.713	0.789
10	0.806	0.159	2.180	0.744	0.857	0.184	3.063	0.820	0.754	0.168	2.201	1.292	0.949	0.194	3.333	0.830
11	0.570	0.105	1.157	0.459	0.609	0.114	1.369	0.508	0.533	0.128	1.221	0.767	0.680	0.117	1.457	0.526
12	0.565	0.109	1.245	0.477	0.585	0.113	1.398	0.524	0.544	0.117	1.204	0.635	0.635	0.113	1.429	0.513
13	0.564	0.113	1.202	0.507	0.598	0.116	1.379	0.519	0.546	0.125	1.227	0.724	0.667	0.119	1.427	0.528
14	0.547	0.111	1.136	0.505	0.565	0.116	1.239	0.516	0.531	0.122	1.137	0.689	0.611	0.118	1.281	0.514
15	0.611	0.115	1.275	0.523	0.627	0.116	1.338	0.512	0.571	0.131	1.236	0.795	0.701	0.119	1.436	0.517

Table 6. RB and RMSE based on different methods under. $\sigma = 0.8$.

	RB															
	GH				TS				Pooled				Fixed			
	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0	b	c	d	a_0
1	0.059	−1.012	−1.001	0.016	0.074	−0.987	−0.983	−0.008	0.165	−0.568	1.856	194.741	0.131	−1.002	−0.993	−0.025
2	0.043	−1.052	−1.069	0.011	0.019	−1.125	−1.073	−0.012	0.221	−0.585	2.171	242.867	0.074	−1.086	−1.056	−0.019
3	0.015	−0.958	−1.031	−0.026	−0.001	−0.956	−0.992	−0.047	0.144	−0.581	2.105	303.912	0.057	−0.970	−1.003	−0.066
4	0.069	−0.948	−1.012	0.058	0.041	−0.944	−0.928	0.052	0.195	−0.575	1.798	205.723	0.104	−0.987	−0.969	0.012
5	0.097	−0.913	−0.997	0.034	0.100	−0.928	−0.958	0.031	0.197	−0.563	1.762	185.937	0.175	−0.960	−0.996	−0.003
6	0.107	−0.366	−0.296	0.051	0.088	−0.465	−0.241	0.068	−0.095	−0.423	0.582	6.234	0.167	−0.514	−0.254	0.044
7	0.105	−0.454	−0.295	0.072	0.059	−0.551	−0.243	0.112	0.003	−0.375	0.655	14.936	0.122	−0.597	−0.235	0.088
8	0.024	−0.504	−0.198	0.049	−0.006	−0.635	−0.068	0.096	−0.125	−0.419	1.116	50.849	0.063	−0.746	−0.119	0.066
9	0.057	−0.622	−0.361	0.054	0.009	−0.748	−0.262	0.088	−0.065	−0.422	1.221	54.991	0.069	−0.856	−0.301	0.061
10	0.097	−0.568	−0.415	0.021	0.055	−0.721	−0.296	0.072	−0.066	−0.411	0.734	23.597	0.127	−0.811	−0.310	0.052
11	0.053	−0.106	0.049	0.073	0.029	−0.183	0.117	0.142	−0.215	−0.404	0.363	3.038	0.103	−0.184	0.131	0.134
12	0.062	−0.066	0.015	0.054	0.002	−0.179	0.160	0.163	−0.127	−0.313	0.289	1.018	0.061	−0.172	0.154	0.146
13	0.077	−0.087	0.013	0.076	0.049	−0.169	0.122	0.163	−0.157	−0.353	0.293	1.278	0.116	−0.166	0.126	0.151
14	0.041	−0.132	0.012	0.057	−0.011	−0.183	0.134	0.138	−0.145	−0.347	0.365	2.348	0.047	−0.179	0.156	0.136
15	0.038	−0.081	0.042	0.055	0.011	−0.168	0.129	0.129	−0.194	−0.369	0.377	2.266	0.079	−0.167	0.181	0.135
	RMSE															
1	2.243	1.194	4.205	2.064	2.333	1.099	4.309	2.067	1.515	0.341	12.102	1785.594	2.584	1.082	4.365	2.112
2	2.355	1.156	3.961	2.178	2.385	1.246	4.390	2.187	1.683	0.345	14.467	2668.937	2.526	1.175	4.328	2.193
3	2.344	1.029	4.587	2.109	2.413	1.007	4.448	2.121	1.599	0.344	13.815	3152.631	2.615	1.047	4.505	2.108
4	2.179	0.928	3.685	2.101	2.224	0.931	3.991	2.103	1.589	0.338	12.948	2218.729	2.415	0.980	4.013	2.079
5	2.352	1.001	3.918	2.121	2.397	1.085	4.274	2.128	1.538	0.337	11.911	1893.848	2.617	1.141	4.355	2.113
6	1.220	0.732	3.114	1.171	1.253	0.784	3.676	1.218	0.893	0.802	4.746	60.884	1.397	0.868	3.807	1.207
7	1.264	0.786	3.387	1.187	1.263	0.867	4.075	1.265	0.968	0.952	5.135	157.956	1.357	0.976	4.133	1.249
8	1.265	1.008	4.064	1.246	1.287	1.062	4.873	1.309	0.965	1.260	6.385	516.319	1.403	1.251	4.987	1.296
9	1.283	1.054	3.826	1.229	1.294	1.203	4.738	1.319	0.975	1.271	7.049	488.901	1.395	1.432	4.917	1.316
10	1.280	1.018	3.683	1.174	1.314	1.117	4.572	1.292	0.921	1.261	5.302	233.565	1.452	1.313	4.743	1.295
11	0.907	0.209	2.571	0.897	0.938	0.211	3.217	1.008	0.763	0.235	2.906	26.355	1.037	0.217	3.375	1.033
12	0.848	0.198	2.185	0.835	0.889	0.208	3.428	1.028	0.764	0.204	2.709	5.643	0.949	0.209	3.383	0.997
13	0.873	0.207	2.381	0.879	0.902	0.211	3.092	1.041	0.761	0.219	2.724	7.330	0.977	0.215	3.190	1.041
14	0.903	0.219	2.506	0.921	0.925	0.224	3.517	1.078	0.772	0.221	2.998	19.253	0.984	0.228	3.615	1.094
15	0.863	0.216	2.413	0.898	0.904	0.221	3.198	1.016	0.736	0.228	2.952	17.777	0.990	0.225	3.367	1.049

From Tables 3–6, we can get some clear conclusions.

(1) For the case without group effects ($\sigma = 0$), the RBs and RMSEs based on all the three models are close to each other, although the true model is the pooled model. Such a result indicates that when the model is misspecified, the proposed model can still fit the data well, and provides reasonable estimates of the model parameters. For estimating the characteristic lifetime a_0, the RMSEs based on the proposed model are almost the same as these based on the true model, while the fixed model performs the worst among the three models in this case.
(2) For the data with group effects ($\sigma = 0.3, 0.5, 0.8$), the proposed model can capture group effects in the data, and the RBs and RMSEs of estimates for the model parameters vary slightly. As a comparison, the RBs and RMSEs based on the pooled model increase significantly as σ increases. Specially, when estimating the characteristic lifetime a_0, neglecting group effects will lead to large biases and variations. Compared with other two models, the performance of the fixed model is a compromise. The RBs and RMSEs based on the fixed model are slightly worse than these based on the proposed model, but much better than these based on the pooled model.
(3) For the GH and TS methods, as we can see from Tables 3–6, regardless of the magnitude of σ, the RBs and RMSEs based on the GH method are almost smaller than the corresponding results based on the TS method. This result is not surprising, because the estimates based on the GH method are the ML estimates, which could have higher efficiency than the TS estimates.

For visualizing these results much better, a box plot is drawn to show the performance of the three models. Figure 3 shows all the point estimates of ϑ based on the three models under the sixth progressive censoring scheme listed in Table 2. The horizontal dashed line in each figure represents the true value, and the red solid point represents the average value of 800 point estimates. For estimating the characteristic lifetime a_0, the GH method based on the proposed model leads to the smallest RMSEs under all the scenarios.

5. Case study

In this section, we will reanalyze the PSALT data of insulating oils listed in Table 1. The lifetime data of the specimens at each group can be obtained through the breakdown voltage divided by the stress rate v_i. As suggested by (Nelson, 2009), we assume the lifetime of each unit follows the Weibull distribution and the relationship between scale parameter and stress satisfies the inverse power law. According to the Kolmogorov-Smirnov test, we find that the test p-value is greater than 0.05, which suggests accepting the null hypothesis: the lifetime data follow a Weibull distribution.

Firstly, three models (the proposed model, pooled model and fixed model) are used to fit the data. The Akaike Information Criterion (AIC) is utilized to select the best model among the three candidates. The results of the parameter estimates and AIC values for different models are listed in Table 7. As can be seen, the AIC values of the proposed model using GH and TS methods are much smaller, which indicates that the propose model fits the data best. The 90% interval estimates of the model parameters are obtained by bootstrap approach, which are shown in Figure 4. From Figure 4, we can see that the lower bound of the 90% interval estimates of σ based on GH is significantly larger than 0.1, which also implies that group-to-group variation exists in the dataset. Figure 5 shows the residuals based on different models. It can be seen that for the proposed model, the residuals for each group are significantly reduced, and they are well balanced among the groups. The sum of squared residuals obtained by GH and TS are 0.216 and 0.212, respectively, which are much lower than the results based on other two models (Pooled: 0.315 and Fixed: 0.311). In addition, we use KW test to verify whether the residual values of each group based on the proposed model are significantly different, and the p-values of the KW test for GH and TS are 0.1729 and

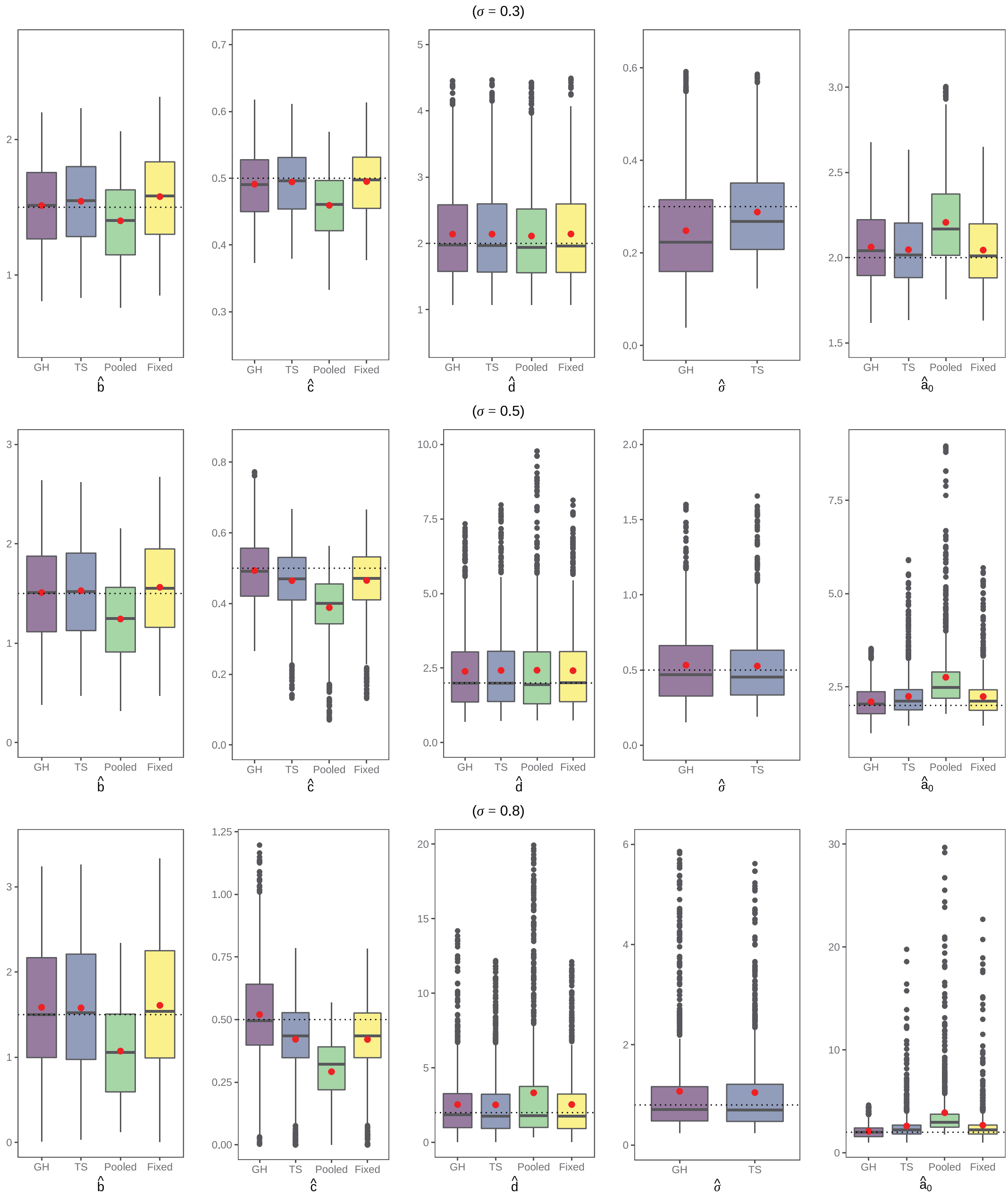

Figure 3. Box plots of point estimates under different σ. The horizontal dash line of each plot denote the true parameter values and the red solid points represent the average value of point estimates.

0.1101, respectively, which are greater than the significance level 0.05. While for the pooled model and the fixed model, the p-values of the KW test are 2.272×10^{-16} and 1.636×10^{-8}, respectively, which means that the heterogeneity still exists among groups. Thus, the proposed model can capture the group effects, and fit the data sufficiently.

When the model has been determined, the estimate of the characteristic lifetime a_0 with normal used condition $s_0 = 30V$ can be obtained. As listed in Table 7, the estimates of a_0 based on the pooled and fixed model are 100.698 and 76.981, respectively, while the estimates based on GH and

Table 7. Model comparison for PSALT data.

		Estimates				
	AIC	$\hat{b}$	$\log(\hat{c})$	$\hat{d}$	$\hat{\sigma}$	$\hat{a}_0$
GH	−523.739	0.772	−53.784	14.800	0.206	96.562
TS	−543.125	0.772	−54.054	14.818	0.211	94.777
Pooled	−459.823	0.771	−54.307	14.890	-	100.698
Fixed	−473.297	0.778	−52.262	14.352	-	76.981

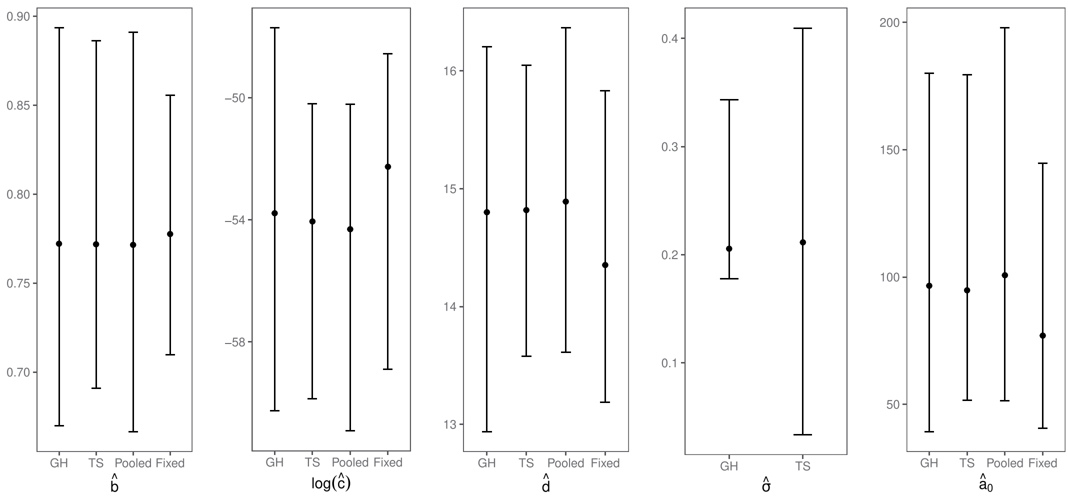

Figure 4. Point estimates (dots) and 90% interval estimates (corresponding lines) for different models.

TS are, respectively, 96.562 and 94.777. As shown in the simulation study, when the group effects exist, the estimate bias is not ignorable for the pooled and fixed models. The AIC and residual analysis have indicated the existence of the group effects. Thus, the estimates of a_0 obtained by GH and TS may be more reasonable.

6. Conclusion and discussion

In this article, we have proposed a new model for PSALT data under progressive censoring with group effects. From a practical point of view, the model can not only consider the heterogeneity among groups, but also provide a flexible censoring scheme for engineers. To infer the point and interval estimation of the model parameters, GH and TS methods are developed. Both approaches are effective to obtain the estimates when the model has group effects. In the simulation studies, we compare the proposed model with other two alternatives, and find that the proposed model can fit the data well whether the data have group effects or not, and also can provide reasonable estimates of the characteristic lifetime. For analyzing the PSALT data of of insulating oils, we find that the heterogeneity is significant among groups, and the proposed model can fit the data well.

The aim of performing PSALT is to predict the characteristic lifetime under the normal used condition, which will directly reflect the reliability of product and affect the formulation of warranty strategy. Therefore, engineers should carefully check the possible group structure of an experiment and incorporate this group effect into their model. (Seo & Pan, 2017) recommended that when analyzing the lifetime data collected by ALT, both the traditional model and the model with group effects can be used for fitting data, then choose a better model based on the results or certain criterion. While, in our simulation, we found that the proposed model performs

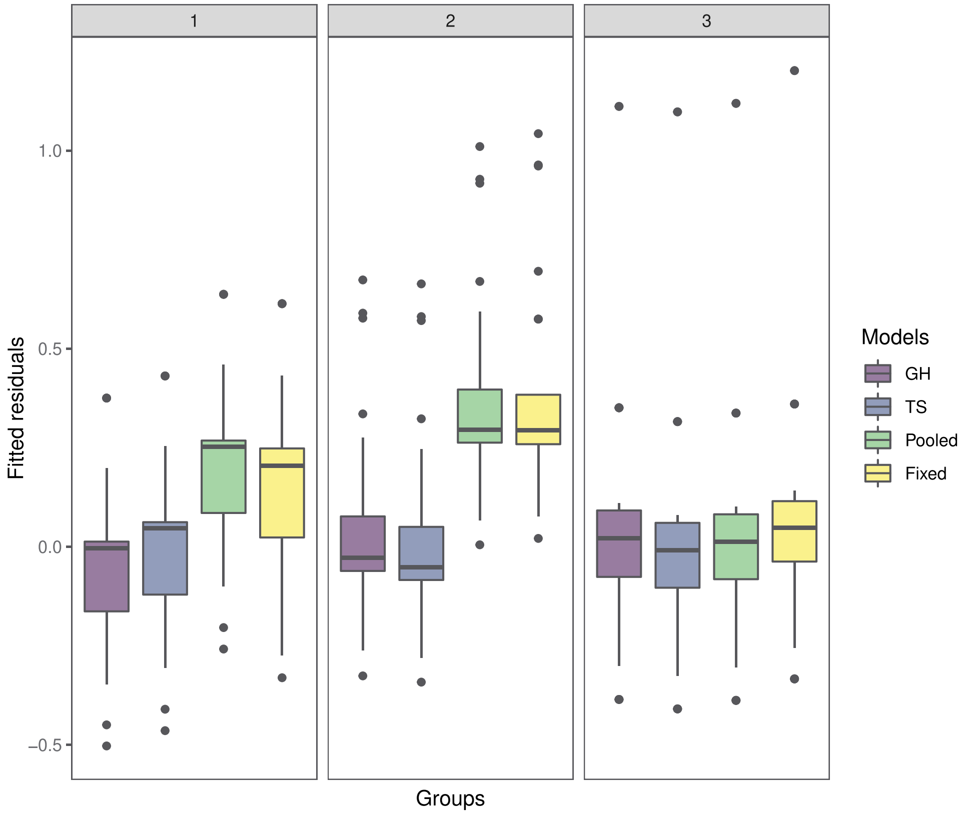

Figure 5. Fitted residuals for different models.

similarly to the pooled model when the group effects do not exist in the data. Thus, an alternative suggestion is that the PSALT data can be analyzed using the proposed models regardless of whether the experiment contains group or cluster structure caused by different raw materials or different test stands, and then judge the existence of group effects according to variance component or performing residual analysis. For engineers, this is a faster and more convenient way to operate the process. However, the model has a drawback, for example, when the group effect is large and the number of groups is small, the performance of the proposed model will be limited. One strategy is to utilize Bayesian methods to analyze the data based on the proposed model. An alternative approach may be using the field data to jointly infer product characteristics under normal used condition (Pan, 2009).

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Appendix

Based on Eq. (10), let $Y_{ij} = \left(t_{ij}/\alpha_i\right)^{\lambda}$, and $f(\lambda)$ be a function of λ:

$$f(\lambda) = \frac{M}{\lambda} + \frac{1}{\lambda}\left[\sum_{i=1}^{k}\sum_{j=1}^{m_i}\log Y_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left(R_{ij}+1\right)Y_{ij}\log Y_{ij}\right],$$

where $M = \sum_{i=1}^{k} m_i$ is the number of all failure units. Based on our assumption that t_{ij} follows Weibull distribution, it can be obtained that $Y_{ij}\sim\exp(1)$. According to the law of large numbers, it can be known that $\sum_{i=1}^{k}\sum_{j=1}^{m_i}\log Y_{ij} \to M\cdot\mathbb{E}\left(\log Y_{ij}\right)$, and $\sum_{i=1}^{k}\sum_{j=1}^{m_i}(R_{ij}+1)Y_{ij}\log Y_{ij} \to N\cdot\mathbb{E}(Y_{ij}\log Y_{ij})$, where $N = \sum_{i=1}^{k} n_i$. Thus, the following results can be obtained: as $N\to\infty$,

$$f(\lambda) \to \frac{(1-\gamma)(M-N)}{\lambda} \quad \text{with probability } 1,$$

where $\gamma = 1 + \int_0^{\infty}\ln(x)e^{-x}\mathrm{d}x$, and $1-\gamma$ is Euler's constant greater than 0. When λ tends to 0^+, we have $\lim_{\lambda\to 0^+} f(\lambda) = +\infty$. When λ tends to positive infinity, we have $\lim_{\lambda\to+\infty} f(\lambda) < 0$. In addition, $f(\lambda)$ is a continuous function. Thus, there is an intersection of $f(\lambda)$ and 0 with probability one.